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Representations of Kronecker forms at higher genus

Master's thesis

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Abstract

Polylogarithms at genus one can be represented as iterated integrals over differential forms generated by a Kronecker function. The corresponding Kronecker form is a representation independent, quasiperiodic one-form with a simple pole. In this thesis, we study Kronecker forms defined on Riemann surfaces of higher genus, leading us to understand the space they span. Our main results include a proof that the dimension of the space of quasiperiodic one-forms with a simple pole is equal to the genus of the underlying manifold, as well as constructive examples of the higher genus Kronecker forms one can use as a basis. Using the language of theta functions, we identify Kronecker forms as ratios of odd theta functions, satisfying a Fay identity. Using Schottky uniformizations, we identify Kronecker forms as averages over the Schottky group, satisfying a procedure for understanding degeneration limits, and corresponding to the components of Enriquez' connection used in an existing generalization of polylogarithms.

Contents

1 Introduction

1.1 Scientific context

A substantial slice of research in the scope of theoretical physics has been devoted to developing an understanding of particle physics and string theory [Sre07,LT89]. One aspect of these developments is in the formalization of scattering amplitude calculations, building upon the underlying mathematics to allow for precision calculations of observables in corresponding theories. On the side of quantum field theories, this is done through the study of Feynman integrals, describing the amplitudes corresponding to processes that particles undergo [Wei22]. On the side of string theories, this is done through the consideration of worldsheet integrals, coming out from the calculations necessary for string amplitudes [SV10].

Over the past few decades, iterated integrals have become a key object in the mathematics underlying calculations of scattering amplitudes. A particular family of iterated integrals concerns itself with repeated logarithmic integrals on the complex plane, or Riemann sphere, known as multiple polylogarithms [Gon01]. These polylogarithms present a compact way to give manageable representations of results of Feynman integrals [Wei07,GSVV10] and worldsheet integrals in string theories [BSS13]. However, these representations are not sufficient to encompass more complicated processes, such as higher loop amplitudes in quantum field theories and one-loop amplitudes in string theories [BBC+22]. The challenges arise from the appearance of underlying manifolds that are of higher dimension, as Calabi–Yau manifolds, or of higher genus, as elliptic and hyperelliptic curves.

Motivated by such calculations, a generalization of multiple polylogarithms on elliptic curves has been formalized and used for calculations. The solutions to Feynman integrals, such as the equal mass sunrise integral [BDDT18, AW18] and kite integral [AW18], have demonstrated the applicability of elliptic polylogarithms to a variety of integrals in particle physics. The elliptic polylogarithms have also been applied towards developing the tools necessary to tackle open and closed-string amplitudes at one loop [BMMS15] and developing procedures for recursive calculations of corresponding integrals [BK21]. A key insight to the formalization used for elliptic polylogarithms is the use of the Kronecker function to generate a tower of integration kernels [BL11], corresponding to a flat connection used to generate polylogarithms. Using a variable that is expressed in the Kronecker function's quasiperiodicity, the Kronecker function is expanded to give the components to differentials that are sufficient for expressing any integral on a complex elliptic curve [BL11]. The Kronecker function is characterized by satisfying a Fay identity [Mat19], an analogue of the partial fraction identity that allows corresponding elliptic polylogarithms to be simplified. Further algebraic relations between polylogarithms can be understood through the connections between the Kronecker function and modular forms [BMS16].

As the scope of objects with an iterated integral representation has grown to encompass those corresponding to genus one surfaces, Feynman integrals and string amplitudes corresponding to higher genus surfaces are now under consideration [BOV21, MMP+23]. A further generalization of elliptic polylogarithms to hyperelliptic curves would allow manageable representations of such objects, and existing approaches to the formalization are already being published [Enr14, EZ21, Ich22, DHS23]. Some approach the question by constructing higher genus flat connections, uniquely identifying one algebraically [Enr14, EZ21], or using convolution relations to construct periodic integration kernels that serve as coefficients [DHS23]. There have also been attempts to generalize the Kronecker function itself [Tsu23]. Building upon this existing work, we seek to approach the question by studying how the Kronecker function may be characterized, generalized to higher genus, and expanded into integration kernels for hyperelliptic polylogarithms.

1.2 Structure

In this thesis, we start by giving a review of the tools used for compact Riemann surfaces at genus one and greater. In Section 2, we review the basics of Riemann surfaces, building up to using theta functions and the Abel map in tandem to give a precise way to define functions with quasiperiodicities and zeros we can control. In Section 3, we explore an approach to working with higher genus Riemann surfaces through the uniformization provided by Schottky covers, which allows functions to be defined as averages of contributions over the group generated by the surface's topology.

After a quick recap of genus zero multiple polylogarithms, we then apply theta functions and Schottky covers to review the construction of the Kronecker function at genus one in Section 4, following the three definitions given by [BL11]. As part of this review, we uniquely characterize the Kronecker function at genus one by its quasiperiodicity and residue, and generalize the Schottky representation from concentric covers [Cha22, Ber88] to ones with an arbitrary generator.

With Section 5, we describe some useful changes to the conventions we use for studying Kronecker functions. Identifying the location of the pole explicitly and including the differential dz in the definition, we study the representation-independent Kronecker form, and characterize it by its quasiperiodicity and unit residue.

In Section 6, we present a new result characterizing the dimensions of spaces of quasiperiodic differentials. In particular, the space of quasiperiodic differentials with a simple pole at a specific location is equal to the genus of the Riemann surface, similar to periodic holomorphic differentials. This means that in searching for Kronecker forms, it is sufficient to find as many independent candidates as the genus of the underlying surface, since these will span the space of quasiperiodic forms with a simple pole.

Finally, in Section 7 and Section 8, we give explicit representations of Kronecker forms using theta functions and Schottky covers.

The representation in terms of theta functions uses an abelianized version of the homotopy group, leaving us with commutative power-counting variables that are easier to work with, but also with a smaller space of differentials. As a consequence of the Fay trisecant identity for theta functions, we find a generalization of the Fay identity to the higher genus Kronecker form and a starting point for an identity for the differentials that it generates.

The representation on the Schottky cover keeps non-commutative power-counting variables, for which we are able to show an explicit expansion matching Enriquez' connection [Enr14]. Although the Kronecker forms on the Schottky cover do not seem to satisfy a Fay identity at higher genera, they do have promising results in other areas, including a convenient way to understand the degeneration of the surface by cutting a cycle.

2 Defining theta functions

Understanding polylogarithms and their generalizations starts with understanding some tools for compact Riemann surfaces. This section will define the concepts relevant to the remainder of the thesis, starting with the topological and differential structure of the surface. This will build up to defining the Abel map, that maps the abstract surface to a set of complex coordinates that may be used to define functions. This will be used to define theta functions, which naturally use these complex coordinates in a way that plays nicely with topological properties of the surface.

For a more detailed account of compact Riemann surfaces and theta functions on them, one may follow [Ber10], which matches the definitions, conventions, and results reviewed in this section.

2.1 Riemann surface and differentials

In this thesis we will be dealing with compact Riemann surfaces $\mathcal{M}$ of genus h. The Riemann surface has local charts U_i such that $\bigcup_i U_i = \mathcal{M}$, with local coordinates $\phi_i : U_i \to \mathbb{C}$.

For a compact manifold $\mathcal{M}$ of genus h, we can identify the fundamental group $\pi_1(\mathcal{M})$ which contains information about homotopically distinct cycles[1] on the surface. In particular, we can identify a canonical basis of $\mathfrak{A}$-cycles and $\mathfrak{B}$-cycles, which we will label $\{\mathfrak{A}_j, \mathfrak{B}_j\}_{j=1}^h$, chosen such that the intersection numbers are

$$\mathfrak{A}_i \# \mathfrak{A}_j = 0, \quad \mathfrak{B}_i \# \mathfrak{B}_j = 0, \quad \mathfrak{A}_i \# \mathfrak{B}_j = \delta_{ij}. \tag{2.1}$$

As we define integration on our surface, it will be useful for us to have a space on which integrations are unambiguously defined without going around extra cycles. We can do this by defining the simply connected domain $\mathcal{L}$, obtained from a dissection of the $\mathcal{M}$ by removing the $\mathfrak{A}$-cycles and $\mathfrak{B}$-cycles and unfolding (see Figure 1). The resulting simply connected domain is bounded by two copies of each cycle.

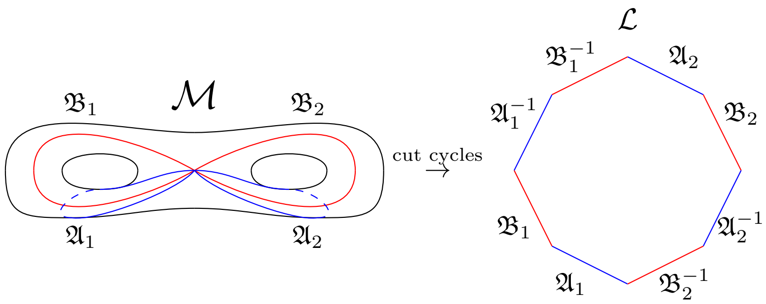

Figure 1: A canonical dissection done at genus two, where the A-cycles (blue) and B-cycles (red) are cut out of the double torus $\mathcal{M}$, leaving an 8-sided simply connected domain $\mathcal{L}$.

Then, we can define a canonical basis of holomorphic Abelian differentials ω_i that are normalized on the $\mathfrak{A}$-cycles, and produce the period matrix τ upon integration on $\mathfrak{B}$-cycles

$$\oint_{\mathfrak{A}_i} \omega_j = \delta_{ij}, \quad \oint_{\mathfrak{B}_i} \omega_j = \tau_{ij}, \tag{2.2}$$

where one can show that the period matrix is symmetric and has positive definite imaginary part[2],

$$\tau_{ij} = \tau_{ji} \quad ; \quad n_i \mathrm{Im}(\tau_{ij}) n_j > 0 \;\forall\; \vec{n} \in \mathbb{R}. \tag{2.3}$$

This period matrix serves a very important role as it contains all of the geometric information of the surface, and we will later use the period matrix for theta functions (Section 2.3), Kronecker functions (Section 4.2), and Kronecker forms (Section 7). To simplify the notation, since we are not considering transformations of the period matrix in this thesis, we will omit it when listing it as a parameter for the corresponding objects. However, it may be useful to study the effect of

[1]That is, elements of the fundamental group, closed curves on the Riemann surface, can be considered equivalent if they can be smoothly deformed into each other. Since the actual coordinates that the curves follow do not matter for this equivalence relation, elements are defined by topological properties, such as the holes or handles of the surface they encircle.

[2]This condition is sometimes denoted by writing $\mathrm{Im}(\tau) > 0$.

transformations of the period matrix in future works, since transformations of τ by the action of $\mathrm{Sp}(2h,\mathbb{Z})$ give a geometrically identical surface (with some $\mathfrak{A}$ and $\mathfrak{B}$ redefined), but lead to representations that converge more quickly.

At genus one, the period matrix corresponds to a single complex number, referred to as the modular parameter. The condition on the positive definite imaginary part becomes simply $\mathrm{Im}(\tau) > 0$. The generators of $\mathrm{Sp}(2,\mathbb{Z})$, $\tau \mapsto \tau + 1$ and $\tau \mapsto -1/\tau$, correspond to identifying $\mathfrak{B}_1 \mapsto \mathfrak{B}_1 + \mathfrak{A}_1$ and $(\mathfrak{A}_1, \mathfrak{B}_1) \mapsto (\mathfrak{B}_1, -\mathfrak{A}_1)$, redefining the generators of the homotopy group through linear transformations.

2.2 The Abel map and Jacobian variety

Using the canonical basis of differentials, we can define the Abel map on the simply connected domain $\mathcal{L}$ (Figure 1),

$$\mathfrak{u} : \mathcal{L} \to \mathbb{C}^h, \quad \mathfrak{u}_i(z) = \int_{p_0}^{z} \omega_i \tag{2.4}$$

where p_0 is an arbitrary basepoint for the integration. As we require our formulas to be independent of the choice of this basepoint, we will find that the Abel map will typically appear only in a difference $\mathfrak{u}_i(z) - \mathfrak{u}_i(x) = \int_x^z \omega_i$.

Since the Abel map is defined with the canonical differentials, it gives us a representation-independent labeling of the points on the manifold. Furthermore, the normalization of the canonical differentials gives us the quasiperiodic properties

$$\mathfrak{u}(z + \mathfrak{A}_j) = \int_{p_0}^{z+\mathfrak{A}_j} \vec{\omega} = \mathfrak{u}(z) + \vec{\delta}_j, \quad \mathfrak{u}(z + \mathfrak{B}_j) = \int_{p_0}^{z+\mathfrak{A}_j} \vec{\omega} = \mathfrak{u}(z) + \tau\vec{\delta}_j, \tag{2.5}$$

which allow us to extend the definition of the Abel map beyond the simply connected domain $\mathcal{L}$, where we introduced $\vec{\delta}_j \in \mathbb{Z}^h$, defined by $(\vec{\delta}_j)_i = \delta_{ij}$. To simplify the notation, the notation can be generalized to using $\mathfrak{u}(\mathfrak{A}_j)$ and $\mathfrak{u}(\mathfrak{B}_j)$ without referencing a point, the integrations over cycles are independent of the starting point.

By taking the target space of the Abel map modulo the periods in Equation 2.5, we identify the Jacobian variety of the Riemann surface $\mathcal{M}$

$$J(\mathcal{M}) = \mathbb{C}^h / (\mathbb{Z}^h + \tau\mathbb{Z}^h), \tag{2.6}$$

where $(\mathbb{Z}^h + \tau\mathbb{Z}^h) = \Lambda(\mathcal{M})$ is the lattice corresponding to the surface. This definition of the Jacobian variety is useful as it makes the Abel map single-valued, quotienting out the aforementioned quasiperiodicities.

2.2.1 Genus one Jacobian variety

The most well known example of the Jacobian variety is the one corresponding to the genus one compact Riemann surface, the torus. The universal cover for the torus is a tiling of the complex plane with parallelograms with edges corresponding to $\mathfrak{A}$ and $\mathfrak{B}$-cycles, where the normalization of $\omega_1(z)$ indicates that we identify $\mathfrak{A} = 1$ and $\mathfrak{B} = \tau$. Taking the quotient of $\mathbb{C}$ with respect to the lattice $\Lambda = \mathbb{Z} + \mathbb{Z}\tau$, we find the the Jacobian variety of the surface $\mathcal{M}$, corresponding to a single parallelogram from the universal cover. Uniquely to genus one, this means that the simply connected domain corresponds exactly to the Jacobian variety of the surface, as seen in Figure 2. This occurs only because the Jacobian variety and the Riemann surface that lives on it are both of dimension 1 at genus one, a correspondence which will be absent at higher genus.

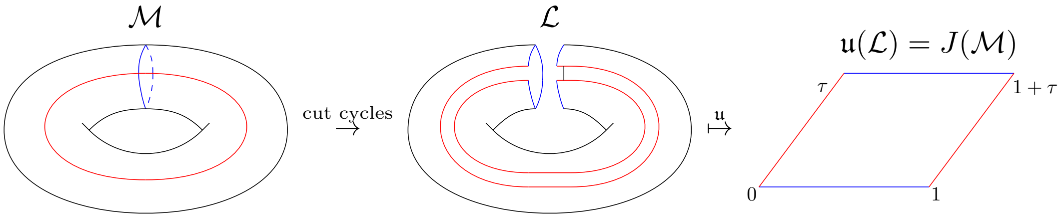

Figure 2: Construction of the Jacobian variety of a torus with modular parameter τ. From left to right we have: the torus labeled $\mathcal{M}$, the simply connected domain after the $\mathfrak{A}$ and $\mathfrak{B}$-cycles are cut, and the image of this simply connected domain after acting with the Abel map with the basepoint at the intersection point of the cycles. The parallelogram formed by this image is precisely the region we get from the Jacobian $J(\mathcal{M})$ from the quotient applied to $\mathbb{C}$.

2.2.2 Genus two Jacobian variety

Let us consider the Jacobian variety corresponding to a Riemann surface of genus two. Though the Jacobian variety is two-dimensional, the Riemann surface $\mathcal{M}$ is only one-dimensional. As a result, we find that $\mathfrak{u}(\mathcal{M}) \subset J(\mathcal{M})$, i.e. the image of the surface of our manifold lives within the variety, as sketched in Figure 3. In the figure, one can see that the boundaries of the simply connected domain $\mathcal{L}$ are transformed by the Abel map into several four-sided boundaries, since we can identify some points on the boundary of $\mathcal{L}$ with each other. Furthermore, the geometrical intuition for how these boundaries produce multiple holes in our Riemann surface is not as simple as the folding of the parallelogram at genus one. Nonetheless, one may imagine identifying the opposite edges of every quadrilateral with each other, producing a hole from each quadrilateral, from which one recovers a more familiar picture like in Figure 1.

This leads to a much more complex picture, since most of the points $\vec{v} \in J(\mathcal{M})$ do not actually lie on the Riemann surface, and the holomorphic differentials present have non-trivial representations that prevent us from restricting ourselves to a smaller space.

2.2.3 Restoring quasiperiodicity

Above, we defined the Jacobian variety as $J(\mathcal{M}) = \mathbb{C}^h/(\mathbb{Z}^h + \tau\mathbb{Z}^h)$, on which $\vec{z}+\mathfrak{A}_j \equiv \vec{z}+\mathfrak{B}_j \equiv \vec{z}$. Though this formulation is useful for understanding the formal aspects of the topology of the surface, it is not the best approach given that the rest of this thesis works with quasiperiodic functions where we must distinguish between points that are a cycle apart.

Instead, for the remainder of this thesis, we will ignore the quotient with respect to the lattice, working directly with $\mathbb{C}^h$ as the parameter space. The image of the manifold with the Abel map allows us to reach a one-dimensional submanifold of the space. As part of this identification, we will use the analytic continuation of the Abel map when moving around cycles, coming up with infinite copies of the surface, just like the universal cover at genus one.

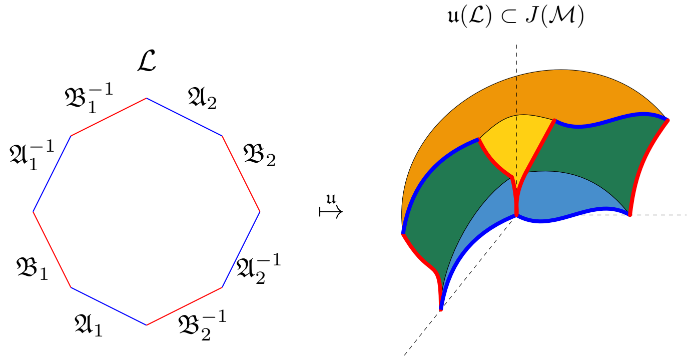

Figure 3: A sketch of the Jacobian variety for a genus two Riemann surface, projected from $\mathbb{C}^2$ to $\mathbb{R}^3$, by $(z_1, z_2) \mapsto (\mathrm{Re}(z_1), \mathrm{Re}(z_2), \mathrm{Im}(z_1) + \mathrm{Im}(z_2))$, so that $\mathfrak{A}$-cycles shift on the horizontal plane and $\mathfrak{B}$-cycles include contributions along the z-axis. The shaded region corresponds to the image of the simply connected domain in the variety, $\mathfrak{u}(\mathcal{L})$, with yellow and orange on the 'outside' and blue and green on the 'inside'. The boundary of this image corresponds to the boundary of the $\mathcal{L}$, with $\mathfrak{A}$ and $\mathfrak{B}$-cycles labeled in blue and red respectively. The point at the origin of the Jacobian variety corresponds to the point at the top / bottom of $\mathcal{L}$; these are equivalent since they are separated by $\mathfrak{u}(\mathfrak{A}_i + \mathfrak{B}_i + \mathfrak{A}_i^{-1} + \mathfrak{B}_i^{-1}) = 0$. This sketch overlooks the fact that the surface must be smooth, and that the surface must meet the boundaries in such a way that copies of the surface would meet smoothly as well.

2.3 Theta functions

The theta function is a key object for working with functions on a compact Riemann surface, providing control over their zeros and periodicities. The standard definition of the theta function is

$$\begin{aligned}\Theta : \mathbb{C}^h &\to \mathbb{C} \\ \vec{z} &\mapsto \sum_{\vec{n} \in \mathbb{Z}^h} \exp(2\pi i [n_j \tau_{jk} n_k / 2 + n_j z_j]).\end{aligned} \tag{2.7}$$

By using the periodicity of the exponential, and relabeling the sum, we find

$$\begin{aligned}\Theta(\vec{z} + \vec{\delta}_j) &= \Theta(\vec{z}), \\ \Theta(\vec{z} + \tau \cdot \vec{\delta}_j) &= \exp(-2\pi i (z_j + \tau_{jj}/2))\Theta(\vec{z}), \\ \Theta(-\vec{z}) &= \Theta(\vec{z}),\end{aligned} \tag{2.8}$$

so the Θ function has some periodicities and quasiperiodicities, and is even.

Looking back at the way the Abel map interacts with cycles, we see that the two quasiperiodicity properties correspond to $\mathfrak{A}$ and $\mathfrak{B}$-cycles,

$$\Theta(\mathfrak{u}(z + \mathfrak{A}_j)) = \Theta(\mathfrak{u}(z)) \quad , \quad \Theta(\mathfrak{u}(z + \mathfrak{B}_j)) = \exp(-2\pi i (\mathfrak{u}(z) + \tau_{jj}/2))\Theta(\mathfrak{u}(z)), \tag{2.9}$$

where $z \in \mathcal{M}$.

2.3.1 Characteristics

We can explore a larger family of theta functions by assigning them characteristics, which shift some of the parameters in the infinite sum, manifesting some modified properties. The definition of a theta function with characteristics $\epsilon, \epsilon' \in \mathbb{C}^h$, is

$$\begin{aligned}\Theta\left[\begin{smallmatrix}\epsilon\\ \epsilon'\end{smallmatrix}\right](\vec{z}) &= \sum_{\vec{n}\in\mathbb{Z}} \exp(2\pi i[(n_j+\epsilon_j)\tau_{jk}(n_k+\epsilon_k)/2 + (n_j+\epsilon_j)(z_j+\epsilon'_j)]) \\ &= \exp(2\pi i[\epsilon_j\tau_{jk}\epsilon_k/2 + \epsilon_j z_j + \epsilon_j\epsilon'_j])\Theta(\vec{z}+\epsilon'/2+\tau\epsilon/2).\end{aligned} \tag{2.10}$$

The characteristics influence the quasiperiodicities of the theta function,

$$\begin{aligned}\Theta\left[\begin{smallmatrix}\epsilon\\ \epsilon'\end{smallmatrix}\right](\vec{z}+\vec{\delta}_j) &= \exp(2\pi i\epsilon_j)\Theta\left[\begin{smallmatrix}\epsilon\\ \epsilon'\end{smallmatrix}\right](\vec{z}), \\ \Theta\left[\begin{smallmatrix}\epsilon\\ \epsilon'\end{smallmatrix}\right](\vec{z}+\tau\vec{\delta}_j) &= \exp(-2\pi i(\epsilon_j + z_j + \tau_{jj}/2))\Theta\left[\begin{smallmatrix}\epsilon\\ \epsilon'\end{smallmatrix}\right](\vec{z}),\end{aligned} \tag{2.11}$$

where the first line corresponds to $\mathfrak{A}$-cycles and the second to $\mathfrak{B}$-cycles.

For half-integer $\epsilon, \epsilon' \in \mathbb{Z}^h/2$, we read the parity property

$$\Theta\left[\begin{smallmatrix}\epsilon\\ \epsilon'\end{smallmatrix}\right](-\vec{z}) = \exp(4\pi i\epsilon_j\epsilon'_j)\Theta\left[\begin{smallmatrix}\epsilon\\ \epsilon'\end{smallmatrix}\right](\vec{z}), \tag{2.12}$$

where for $\epsilon = \epsilon' = 0$ we recover the even theta function Θ.

Finally, the characteristics are related up to their real integer part,

$$\Theta\left[\begin{smallmatrix}\epsilon+\nu\\ \epsilon'+\nu'\end{smallmatrix}\right](\vec{z}) = \exp(2\pi i\epsilon_j\nu'_j)\Theta\left[\begin{smallmatrix}\epsilon\\ \epsilon'\end{smallmatrix}\right](\vec{z}),\ \forall \nu,\nu' \in \mathbb{Z}^h, \tag{2.13}$$

which is particularly useful in the case that $\epsilon, \epsilon' \in \mathbb{Z}^h/2$ since it means we can restrict ourselves to $\epsilon, \epsilon' \in \{0, 1/2\}^h$.

In particular, when $4\epsilon_j\epsilon'_j$ is odd, our theta function itself is odd (Equation 2.12), which manifestly gives us $\Theta\left[\begin{smallmatrix}\epsilon\\ \epsilon'\end{smallmatrix}\right](0) = 0$. Taking into account that Equation 2.13 restricts half-integer characteristics to $\epsilon, \epsilon' \in \{0, 1/2\}^h$, we have 2^{2h} possible half-integer characteristics, of which $2^{h-1}(2^h - 1)$ are odd.

It is common to avoid writing out the full characteristics of theta functions, such as denoting the odd theta function at genus one as $\theta_1(z) = \Theta\begin{bmatrix}1/2\\1/2\end{bmatrix}$, or using a subscript to contain the characteristics of the theta function. In this thesis, we restrict the use of subscripts to the way they are defined in Section 2.4.2 to work with vectors or divisors. Instead, due to their prevalence in the discussion of Kronecker functions and Kronecker forms in future sections, we will in general write θ to indicate a consistently chosen, but arbitrary, odd theta function,

$$\theta \equiv \Theta\left[\begin{smallmatrix}\epsilon\\ \epsilon'\end{smallmatrix}\right] \text{ for some } \epsilon, \epsilon' \in \mathbb{Z}^h/2, \text{ such that } 4\epsilon_j\epsilon'_j \text{ odd.} \tag{2.14}$$

2.3.2 Prime form

An object that comes up often in the study of compact Riemann surfaces, and a few times in this thesis, is the prime form. It is defined as

$$E(z,x) = \frac{\theta(\mathfrak{u}(z) - \mathfrak{u}(x))}{\psi(z)\psi(x)}, \tag{2.15}$$

where θ is an arbitrary odd theta function, and ψ is a half-differential defined as

$$\psi(z) = \sqrt{\sum_{j=1}^{h} \partial_{v_j}\theta(\vec{v})\Big|_{\vec{v}=\vec{0}}\omega_j(z)} = \sqrt{d_z\Big(\theta(\mathfrak{u}(z) - \mathfrak{u}(p))\Big)\Big|_{p=z}}. \tag{2.16}$$

Using properties of the theta function, one can show that the prime form satisfies

- Antisymmetry: $E(z,x) = -E(x,z)$,
- $\mathfrak{A}$ Periodicity: $E(z+\mathfrak{A}_j, x) = E(z,x)$,
- $\mathfrak{B}$ Quasiperiodicity: $E(z+\mathfrak{B}_j, x) = \exp(-2\pi i[\tau/2 + \mathfrak{u}(x) - \mathfrak{u}(z)])E(z,x)$,
- Vanishing on the diagonal: $E(z,x) = \frac{(z-x)}{\sqrt{dz}\sqrt{dx}}(1 + O((z-x)^2))$.

Furthermore, the prime form is independent of the characteristics of the odd theta function chosen; this can be proven by noticing that it has the same periodicities and divisor no matter what characteristics are chosen.

2.4 Divisors

A very important topic when discussing functions on compact Riemann surfaces is the idea of a divisor. Divisors are defined as formal sums of points with integer coefficients

$$D = \sum_i n_i P_i, \tag{2.17}$$

where the degree of a divisor is the sum of the coefficients, that is $\deg(D) = \sum_i n_i$. In some of the future sections, we will write D_k to indicate that we are speaking about a divisor of degree k.

Divisors are useful for keeping track of the zeros and poles of functions, identifying the coefficients as the orders of the zeros, using positive coefficients, and the orders of the poles, using negative coefficients. For example, the divisor of the meromorphic function $1/z$ on the Riemann sphere is $D = (\infty) - (0)$ as a formal sum, and the degree of the divisor is $\deg(D) = 0$. Indeed, it is possible to show that any single-valued, and consequently periodic, meromorphic function on $\mathcal{M}$ has a divisor of degree zero.

However, the theta functions we are working with are quasiperiodic, so this result does not apply. Indeed, the theta functions have no poles, and so for the remainder of this thesis, we will only refer to strictly positive divisors, for which $n_i > 0$. With this in mind, we will use the concept of the divisor only to study more carefully the vanishing locus of theta functions as they come up (Section 2.4), and the divergences those theta functions have for the corresponding Kronecker functions (Section 4.2 and Section 7).

In order to make this connection to theta functions, we will rely on the Abel map of a divisor, defined simply as

$$\mathfrak{u}\left(\sum_i n_i P_i\right) = \sum_i n_i \mathfrak{u}(P_i). \tag{2.18}$$

2.4.1 Jacobi inversion theorem

Though there are many properties that come out of understanding the structure of the Abel map on the Jacobian variety[3], the result useful to us in understanding the divisor of the theta function is the Jacobi inversion theorem.

The Jacobi inversion theorem states that for every $\vec{z} \in J(\mathcal{M})$ we can find a divisor D_h of degree h such that $\mathfrak{u}(D) = \vec{z}$. With few exceptions, known as special divisors, the choice of the divisor corresponding to a particular vector is unique. In future sections, we will assume that

[3]Some of those not mentioned are that the Abel map is an immersion and embedding of $\mathcal{M}$ into $J(\mathcal{M})$, as well as the Abel theorem relating divisors of meromorphic functions with vanishing Abel maps [Ber10].

none of the divisors we work with are special. This gives us a precise isomorphism between the h-dimensional variety that the Riemann surface lives in, and the h-dimensional variety spanned by h points that live on it. In the upcoming section, we will use this knowledge to draw connections between the vectors in $\mathbb{C}^h$ that theta functions use as arguments and the corresponding divisors.

2.4.2 Divisors of theta functions

Though the odd theta functions above are sufficient to find functions that vanish at the origin, we can develop even more control over the vanishing locus of the theta function by studying its divisor. Writing $\theta_{\vec{v}}(\mathfrak{u}(z)) = \Theta(\mathfrak{u}(z) - \vec{v})$, one can show [Ber10, Proposition 5.2.2] that the divisor D_h of $\theta_{\vec{v}} \circ \mathfrak{u}$ satisfies $\mathfrak{u}(D_h) = \vec{v} - \vec{K}$, where $\vec{K}$ is the vector of Riemann constants

$$K_j = \frac{\tau_{jj}}{2} - \sum_{k=1}^{h} \int_{p_0}^{z=p_0+\mathfrak{A}_k} \mathfrak{u}_j(z)\omega_k(z), \tag{2.19}$$

where p_0 is the basepoint of the Abel map. Using the Jacobi inversion theorem, we recognize that this divisor D_h must be made out of h points, since it corresponds to an arbitrary vector in $\mathbb{C}^h$. Therefore, theta functions composed with the Abel map at genus h have precisely h zeros.

Rewriting the statement above using $\vec{v} = \mathfrak{u}(D_h) + \vec{K}$, we see that $\Theta(\mathfrak{u}(z) - \mathfrak{u}(D_h) - \vec{K})$ vanishes when z is a point in D. Without loss of generality, we find that $\Theta(-\mathfrak{u}(D_{h-1}) - \vec{K})$ vanishes for any divisor D_{h-1} of degree $h-1$, which is sometimes used as the defining feature of the vector of Riemann constants [EF00].

Instead of using the vector to define theta functions, we can identify a theta function directly from its divisor, writing

$$\theta_{D_h}(\mathfrak{u}(z)) = \Theta(\mathfrak{u}(z) - \mathfrak{u}(D_h) - \vec{K}), \tag{2.20}$$

which as a function of z has the divisor D_h. When searching for theta functions that vanish at the origin, we can thus go beyond the odd theta functions suggested in Section 2.3.1. Choosing $D_h = p_0 + D_{h-1}$, the corresponding theta function satisfies

$$\theta_{D_h}(\mathfrak{u}(p_0)) = 0 \implies \theta_{D_h}(\vec{0}) = 0. \tag{2.21}$$

Having fixed the point p_0, the remaining $h-1$ points contained in D_{h-1} indicate that this family of theta functions that vanish at the origin correspond to an $(h-1)$-dimensional variety.

Conversely, a careful derivation tells us that the theta function's vanishing locus is an $(h-1)$-dimensional variety [Ber10, Proposition 5.2.3]

$$\Theta(\vec{e}) = 0 \iff \vec{e} = \mathfrak{u}(D_{h-1}) + \vec{K}, \tag{2.22}$$

which can be generalized to arbitrary theta functions as

$$\theta_{\vec{v}}(\vec{e}) = 0 \iff \vec{e} - \vec{v} = \mathfrak{u}(D_{h-1}) + \vec{K}. \tag{2.23}$$

At genus one, this tells us that theta functions vanish at a single point. This is consistent with what we would expect, since we know from Section 2.2.1 that the Jacobian variety is the same dimension at the cover, so the single point corresponds precisely to the degree 1 divisor that defines the function. This special case at genus one uniquely allows us to identify that the pole that $1/\theta(\alpha)$ has on the simply connected domain is only at $\alpha = 0$, and can be cancelled a linear term. Thus, $\alpha/\theta(\alpha)$ is holomorphic near $\alpha = 0$, which will prove to be useful as we consider the genus one Kronecker function in Section 4.2.

At higher genus, however, theta functions vanish on some submanifold of $\mathbb{C}^h$, making them more challenging to work with. These difficulties are related to those we encountered when describing the image of the Abel map in the Jacobian variety in Section 2.2.2, as there is a large step up in the complexity of the analytic structure of the manifold as we step to higher genera. A theta function vanishing at the origin will also vanish on a submanifold passing through it, preventing us from cancelling the divergence by a term as simple as at genus one, a problem we will deal with as we generalize the Kronecker function to higher genus in Section 7.3.1.

3 Defining Schottky covers

Schottky covers allow us to have a precise representation for Riemann surfaces of genus greater than 1. This allows us to numerically verify the identities that we derive, as well as make actual calculations of the kernels and their iterated integrals. More importantly, it provides a mathematical framework for working with the non-commutative homotopy of the compact Riemann surface[4].

For the convenience of the reader, the definitions and most important tools of a Schottky cover will be explained here. A complete account of Schottky covers and the conclusions they allow us to make about Riemann surfaces is available in [Bob11].

3.1 Schottky cover definition

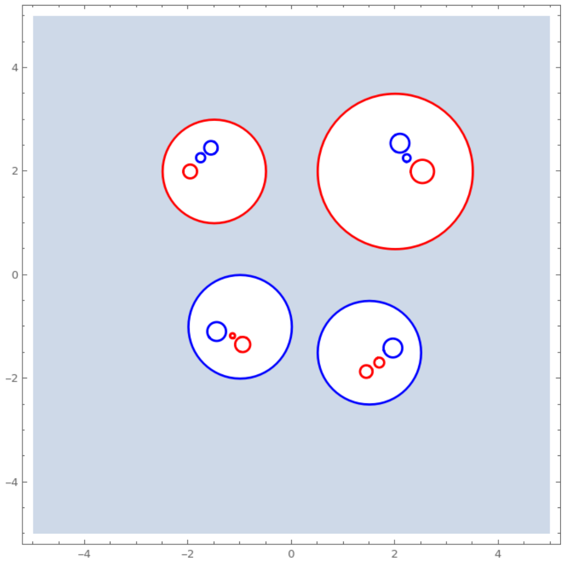

Figure 4: A genus two Schottky cover. The shaded region is the fundamental domain $\mathcal{L}$, lying outside the disks defining the Schottky cover. Each pair of disks of the same color corresponds to a pair of $\mathfrak{A}$-cycles that are identified with each other. Each of the large circles contains the images of the three other circles when acting with the corresponding generator.

[4]We will find that on the Schottky cover, shifting a point by $\mathfrak{B}_j$ and $\mathfrak{B}_k$ will correspond to acting with non-commuting Moebius transformations γ_j and γ_k, where $\gamma_j\gamma_k z \neq \gamma_k\gamma_j z$. This is unlike the case on the Jacobian variety, where with the Abel map we had $\mathfrak{u}(z + \mathfrak{B}_j + \mathfrak{B}_k) = \mathfrak{u}(z + \mathfrak{B}_k + \mathfrak{B}_j)$.

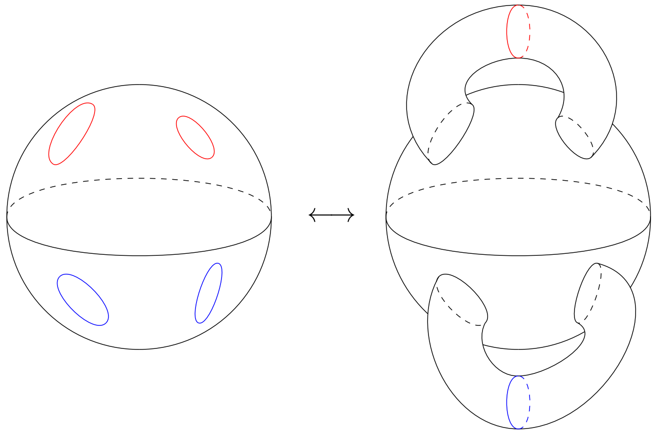

Figure 5: A sketch of the Riemann sphere corresponding to the Schottky cover in Figure 4. Identifying the circles with each other and deforming the surface such that they meet reveals the handles of the corresponding genus two surface, and the identification of the circles as $\mathfrak{A}$-cycles. Moving into a circle on the left corresponds to moving around a handle on the right, which for the purpose of multivalued functions brings the point to a copy of the fundamental domain.

Recall that Moebius transformations on the complex plane, $\mathrm{PSL}(2,\mathbb{C})$, can be written as $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, with $ad - bc = 1$, acting on points as

$$Mz = \frac{az+b}{cz+d}, \tag{3.1}$$

with their composition satisfying matrix multiplication, $M_1(M_2 z) = (M_1 M_2)z$. Similarly to how translation by $\vec{\delta}_j$ and $\tau\vec{\delta}_j$ were used to correspond to $\mathfrak{A}$ and $\mathfrak{B}$-cycles on the Jacobian variety (Equation 2.5), we will use Moebius transformations to construct a cover on which we can understand the Riemann surface.

Consider h disjoint pairs of discs $\{D_j, D'_j\}$ with interiors $\{\mathring{D}_j, \mathring{D}'_j\}$. We can identify Moebius transformations $\gamma_j \in \mathrm{PSL}(2,\mathbb{C})$ that map the exterior of D_j to the interior of D'_j, and mapping the boundary of D_j to the boundary of D'_j,

$$\begin{aligned} \gamma_j(\bar{\mathbb{C}} \setminus \mathring{D}_j) &= D'_j, \\ \gamma_j(\partial D_j) &= \partial D'_j, \end{aligned} \tag{3.2}$$

where $\bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.

These Moebius transformations can be treated as the generators of a Schottky group Γ, a finitely generated subgroup of $\mathrm{PSL}(2,\mathbb{C})$. We can identify a fundamental domain[5] $\mathcal{L} = \bar{\mathbb{C}} \setminus \bigcup_{j=1}^{h}(\mathring{D}_j \cup \mathring{D}'_j)$, containing a single 'copy' of our manifold. Applying the Schottky group to

[5]Note that despite using the same notation as the simply connected domain $\mathcal{L}$ from the previous section (e.g. Figure 1), the fundamental domain is not simply connected. It is however similar in that it contains a single copy of our manifold for the purpose of tracking quasiperiodicity, and one could in principle cut out appropriate curves corresponding to $\mathfrak{B}$-cycles to recover a simply connected region.

the fundamental domain we identify the covering $\Omega = \bigcup_{\gamma \in \Gamma} \gamma(\mathcal{L})$, where we recognize the genus h Riemann surface $\mathcal{M} = \Omega/\Gamma$. The boundaries of the discs D_j and D_j' can be interpreted as $\mathfrak{A}$-cycles and the action of the generators γ_j can be interpreted as $\mathfrak{B}$-cycles. By construction, functions with no monodromy around $\mathfrak{A}$-cycles become single-valued on this cover.

For these generators, we can identify corresponding fixed points P_j and P_j' such that

$$\begin{aligned} \gamma_j(P_j) = P_j; \quad & \gamma_j(P_j') = P_j' \\ \lim_{n\to\infty} \gamma_j^n Q = P_j; \quad & \lim_{n\to\infty} (\gamma_j^{-1})^n Q = P_j \quad \forall Q \notin \{P_j, P_j'\}. \end{aligned} \tag{3.3}$$

3.1.1 Concentric Schottky cover

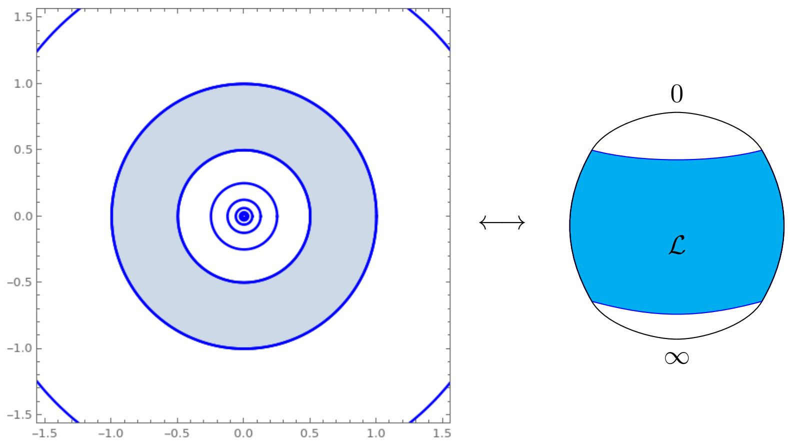

Figure 6: A genus one Schottky cover, chosen such that the circles are concentric, with nome $q = 0.5$. The shaded region is the fundamental domain $\mathcal{L}$. The $\mathfrak{A}$-cycles are marked with blue contours. The left image depicts the Schottky cover on the complex plane, with images of the $\mathfrak{A}$-cycles under the action of the Schottky group also marked in blue. The right image depicts the Schottky cover on the corresponding Riemann sphere, demonstrating how the 'inside' of the outer circle contains infinity.

The simplest example of a Schottky cover is the concentric Schottky cover used at genus one. In this case, we identify two circles centered at zero with radii 1 and $|q|$, with $|q| < 1$. Formally, we identify the inside of the circle with radius $|q|$ as $\{z \in \bar{\mathbb{C}} : |z| < |q|\}$ containing 0, and the inside of the circle with radius 1 as $\{z \in \bar{\mathbb{C}} : |z| > 1\}$ containing ∞. Then, the generator of the Schottky group, $\gamma_1 : z \mapsto qz$, correctly maps the outside of one circle to the inside of the other.

In this case, $\omega_1 = \frac{1}{2\pi i}\frac{dz}{z}$, and we find that the modular parameter is

$$\tau = \int_x^{\gamma_1 x} \omega_1 = \frac{1}{2\pi i}[\ln(qx) - \ln(x)] = \frac{\ln(q)}{2\pi i} + n, \tag{3.4}$$

for $n \in \mathbb{Z}$, where this ambiguity by an integer can be ignored[6]. This allows us to identify $q = e^{2\pi i \tau}$, commonly refered to as the nome of the genus one surface, and a common parameter used for expansions of functions since $\text{Im}(\tau) > 0 \implies |q| < 1$ ensures convergent power series.

[6]It corresponds to the transformation $\tau \mapsto \tau + 1$, one of the generators of the $\text{Sp}(2, \mathbb{Z})$ symmetry group of the modular parameter. Since Schottky covers identify $\mathfrak{A}$-cycles with circles, there is an ambiguity for adding extra $\mathfrak{A}$-cycles to the modular parameter.

3.2 Holomorphic differential basis

In order to have well-defined sums over elements of the group, we will need to define cosets

$$\begin{aligned} \Gamma/\Gamma_i &= \{\gamma_{j_1}^{n_1}\cdots\gamma_{j_k}^{n_k} : \gamma_{j_k} \neq \gamma_i\}, \\ \Gamma_i\backslash\Gamma &= \{\gamma_{j_1}^{n_1}\cdots\gamma_{j_k}^{n_k} : \gamma_{j_1} \neq \gamma_i\}, \end{aligned} \tag{3.5}$$

which don't end (Γ/Γ_i) or don't start ($\Gamma_i\backslash\Gamma$) with a particular generator or its inverse.

Using these cosets, we can identify the canonical holomorphic differentials corresponding to the Schottky group

$$\begin{aligned} \omega_i(z|\Gamma) &= \frac{1}{2\pi i}\sum_{\gamma\in\Gamma/\Gamma_i}\left(\frac{1}{z-\gamma(P_i')} - \frac{1}{z-\gamma(P_i)}\right)dz \\ &= \frac{1}{2\pi i}\sum_{\gamma\in\Gamma_i\backslash\Gamma}\left(\frac{1}{\gamma(z)-P_i'} - \frac{1}{\gamma(z)-P_i}\right)d(\gamma(z)), \end{aligned} \tag{3.6}$$

which can then be used to define the Abel map

$$\mathfrak{u}_i(p|\Gamma) = \int_{p_0}^{p}\omega_i = \frac{1}{2\pi i}\sum_{\gamma\in\Gamma/\Gamma_i}\ln\{p, \gamma(P_i'), p_0, \gamma(P_i)\}$$

where we use the cross-ratio $\{a,b,c,d\} = \frac{(a-b)(c-d)}{(a-d)(c-b)}$. One way to recognize these differentials is by recalling the normalization conditions they must satisfy when integrating along the $\mathfrak{A}$-cycles. The circle corresponding to $\mathfrak{A}_j$ contains the fixed point P_j, as well as both γP_k and $\gamma P_k'$ for $\gamma = \gamma_j\cdots$. Integrating along the circle, we can use the Cauchy residue theorem, where the residues from γP_k and $\gamma P_k'$ cancel each other. The only contribution that will not cancel out is when integrating ω_j since it includes $dz/(z-P_j)$, giving the unit normalization we desire.

Since the Abel map is made out of cross-ratios of the points, it is invariant under Moebius transformations, where we transform each point as $z \mapsto Mz$ and elements of the group as $\gamma \mapsto M\gamma M^{-1}$,

$$\{Ma, Mb, Mc, Md\} = \{a,b,c,d\} \implies \mathfrak{u}_i(Mp|M\Gamma M^{-1}) = \mathfrak{u}_i(p|\Gamma). \tag{3.7}$$

This means that the period matrix is preserved under these transformations too, implying that Schottky covers related by a Moebius transformation represent the same Riemann surface. Thus, when constructing differentials on the surface, which are independent of the representation, we must be careful to construct them in a way that is Moebius invariant. In order to do this, we can take a look at our fundamental building blocks and see how they transform

$$\begin{aligned} d(Mz) &= d\left(\frac{az+b}{cz+d}\right) = \frac{a(cz+d)-c(az+b)}{(cz+d)^2}dz = \frac{1}{(cz+d)^2}dz, \\ (Mz-Mx)^n &= \left(\frac{az+b}{cz+d} - \frac{ax+b}{cx+d}\right)^n = \frac{1}{(cz+d)^n(cx+d)^n}(z-x)^n. \end{aligned} \tag{3.8}$$

Indeed, with these properties in mind, we can verify that each term in the expansion of $\omega_i(z|\Gamma)$ is Moebius invariant[7],

$$\frac{d(Mz)(MP_i' - MP_i)}{(Mz-MP_i)(Mz-MP_i')} = \frac{dz(P_i'-P_i)}{(z-P_i)(z-P_i')}. \tag{3.9}$$

As we make definitions for one-forms in future sections, we will write them using analogous Moebius invariant terms to make them independent of the choice of Schottky cover.

[7] In some sense, one may interpret these terms as cross-ratios with a 'repeated' variable making the differential.

4 Polylogarithms and the Kronecker function at low genera

With the tools available to us on compact Riemann surfaces, we review existing definitions of multiple polylogarithms at genus zero [Gon01] and genus one [BL11]. The genus zero polylogarithms serve as a starting point for understanding what polylogarithms are and what properties we desire from the integration kernels. The genus one polylogarithms are the inspiration for the generalizations the remainder of the thesis proposes. Unlike the genus zero case, where only a single integration kernel is necessary, the genus one construction relies on a generating function for an infinite number of kernels. This generating function, known as the Kronecker function, will be the central focus of our analysis.

4.1 Polylogarithms at genus zero

Multiple polylogarithms are defined as

$$G(a_1, a_2, \cdots, a_n; z) = \int_0^z \frac{dt}{t - a_1} G(a_2, \cdots, a_n; t) \tag{4.1}$$

where the length 0 polylogarithm is normalized $G(; z) = 1$, except when the argument z coincides with the basepoint $G(\vec{a}; 0) = G(; 0) = 0$. The label $\vec{a}$ controls the locations of the poles of the integration kernels.

Due to their definition as iterated integrals, multiple polylogarithms satisfy a shuffle relation

$$G(a_1, \cdots, a_r; z)G(a_{r+1}, \cdots, a_{r+s}; z) = \sum_{\sigma \in \sum(r,s)} G(a_{\sigma(1)}, \cdots a_{\sigma(r+s)}; z) \tag{4.2}$$

where $\sum(r, s)$ is a subgroup of the permutation group S_{r+s} acting on $\{a_1, \cdots, a_{r+s}\}$ that leaves the order of the elements $\{a_1, \cdots, a_r\}$ and $\{a_{r+1}, \cdots, a_{r+s}\}$ unchanged.

We also find that the integration kernels present in the polylogarithms satisfy a partial fraction identity

$$\frac{1}{t - a} \frac{1}{\tilde{t} - a} = \frac{1}{t - a} \frac{1}{\tilde{t} - t} + \frac{1}{\tilde{t} - a} \frac{1}{t - \tilde{t}}, \tag{4.3}$$

which allows one to remove the argument of the polylogarithm from the label [BSS13, Equation 5.21].

4.2 Polylogarithms and the Kronecker function at genus one

Unlike the genus zero case, where only a single integration kernel was necessary, polylogarithms at genus one require an infinite tower of integration kernels. These kernels $g^{(m)}$ are conveniently generated by the Kronecker function $F : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ as

$$\alpha F(z, \alpha) = \sum_{m=0}^{\infty} g^{(m)}(z) \alpha^m, \tag{4.4}$$

where $F(z, \alpha)$ and $g^{(m)}(z)$ are functions of $z \in \mathbb{C}$, with α as a power-counting variable, and m indicating the corresponding weight of the kernel $g^{(m)}$. The expansions for the kernels are known, with the lowest weights being

$$g^{(0)}(z) = 1, \tag{4.5}$$

$$g^{(1)}(z) = \pi \cot(\pi z) + 4\pi \sum_{m=1}^{\infty} \sin(2\pi m z) \sum_{n=1}^{\infty} q^{mn}, \tag{4.6}$$

$$g^{(2)}(z) = -2\zeta_2 + 8\pi^2 \sum_{m=1}^{\infty} \cos(2\pi m z) \sum_{n=1}^{\infty} n q^{mn}, \tag{4.7}$$

where $q = \exp(2\pi i\tau)$. The only kernel containing a pole at $z = 0$ is $g^{(1)}$, serving as an analogue of $1/z$.

With these integration kernels, indexed by their weight m, we can define multiple elliptic polylogarithms as

$$\tilde{\Gamma}(\begin{smallmatrix} m_1 & m_2 & \cdots & m_n \\ a_1 & a_2 & \cdots & a_n \end{smallmatrix}; z) = \int_0^z dt\; g^{(m_1)}(t - a_1)\Gamma(\begin{smallmatrix} m_2 & \cdots & m_n \\ a_2 & \cdots & a_n \end{smallmatrix}; t). \tag{4.8}$$

These polylogarithms satisfy many properties analogous to those at genus zero [BL11, BDDT18], though a full description is beyond the scope of this thesis.

Focusing on the Kronecker function, we can uniquely identify it[8] as a function of $z \in \mathbb{C}$ and a formal variable α, with the quasiperiodicity and single simple pole

$$\begin{gathered} F(z+1, \alpha) = F(z, \alpha) \quad ; \quad F(z+\tau, \alpha) = e^{-2\pi i \alpha} F(z, \alpha), \\ \mathrm{res}_{z=0} F(z, \alpha) = 1. \end{gathered} \tag{4.9}$$

Applying the expansion of the Kronecker function to the integration kernels (Equation 4.4), we find that the above properties translate to

$$\begin{gathered} g^{(m)}(z+1) = g^{(m)}(z) \quad ; \quad g^{(m)}(z+\tau) = \sum_{k=0}^{m} (-2\pi i)^k g^{(m-k)}(z), \\ \mathrm{res}_{z=0} g^{(m)}(z) = \delta_{m1}. \end{gathered} \tag{4.10}$$

Identifying z as a coordinate on the universal cover of our torus with period matrix τ, we can recognize the quasiperiodicities of the Kronecker function as corresponding to $\mathfrak{A}$ and $\mathfrak{B}$-cycles.

Furthermore, the Kronecker function satisfies a Fay identity [BL11]

$$F(z, \alpha)F(\tilde{z}, \tilde{\alpha}) = F(z, \alpha + \tilde{\alpha})F(\tilde{z} - z, \tilde{\alpha}) + F(\tilde{z}, \alpha + \tilde{\alpha})F(z - \tilde{z}, \alpha), \tag{4.11}$$

which can be verified by studying the residues and quasiperiodicity of both sides of the equation. Aside from the Kronecker function being characterized as a fundamental solution of the Fay identity [Mat19], it implies an identity for the integration kernels [BSS13]

$$\begin{aligned} g^{(m)}(z_1)g^{(n)}(z_2) = &(-1)^{n-1} g^{(m+n)}(z_1 - z_2) \\ &+ \sum_{r=0}^{n} \binom{m+r-1}{r} g^{(m+r)}(z_1) g^{(n-r)}(z_2 - z_1) \\ &+ \sum_{r=0}^{m} \binom{n+r-1}{r} g^{(n+r)}(z_2) g^{(m-r)}(z_1 - z_2), \end{aligned} \tag{4.12}$$

[8] This is shown in Section 6, and is only true at genus one. At higher genera, analogous constraints leave us with a larger space of functions.

a proof of which is contained in Appendix A. This Fay identity for the kernels serves as an analogue to the partial fraction identity (Equation 4.3) at genus zero.

With all these properties in mind, let us move on to a few definitions given for the Kronecker function [BL11].

4.2.1 Theta function representation

The first and most commonly given definition of the Kronecker function is in terms of the unique odd theta function at genus one, which we simply denote θ (Equation 2.14),

$$F(z, \alpha) = \frac{\theta(z+\alpha)\theta'(0)}{\theta(z)\theta(\alpha)}. \tag{4.13}$$

Since at genus one, the universal cover is identical to the Jacobian variety, we can omit the Abel map that the theta function typically uses for inputs of points. Using the properties of the odd theta function, Verifying the defining properties of this Kronecker function

$$\begin{aligned} \theta(v+1) = \theta(v) &\implies F(z+1, \alpha) = F(z, \alpha), \\ \theta(v+\tau) = e^{-2\pi i v}\theta(v) &\implies F(z+\tau, \alpha) = \frac{e^{-2\pi i(z+\alpha)}}{e^{-2\pi i z}} F(z, \alpha) = e^{-2\pi i \alpha} F(z, \alpha), \\ \theta(z) \simeq \theta'(0)z + O(z^3) &\implies \mathrm{res}_{z=0} F(z, \alpha) = 1. \end{aligned} \tag{4.14}$$

Furthermore, the theta function representation is the original source of the Fay identity of the Kronecker function [Mat19], since the theta functions are known to satisfy the Fay trisecant identity [Mum84][9]

$$\begin{aligned} &\theta(z+\alpha)\theta(\tilde{z}+\tilde{\alpha})\theta(z-\tilde{z})\theta(\alpha+\tilde{\alpha}) = \\ &\quad \theta(z+\alpha+\tilde{\alpha})\theta(z-\tilde{z}-\tilde{\alpha})\theta(\tilde{z})\theta(\alpha) + \\ &\quad \theta(\tilde{z}+\alpha+\tilde{\alpha})\theta(z-\tilde{z}+\alpha)\theta(z)\theta(\tilde{\alpha}) \end{aligned} \tag{4.15}$$

which implies after division by $\theta(z-\tilde{z})\theta(\alpha+\tilde{\alpha})\theta(\tilde{z})\theta(\alpha)\theta(z)\theta(\tilde{\alpha})$ and multiplication by $\theta'(0)$ the Fay identity for the Kronecker functions (Equation 4.11).

4.2.2 Schottky cover representation

In [BL11], we read the definition

$$F(z, \alpha) = -2\pi i \left(\frac{v}{1-v} - \frac{w}{1-w} + \sum_{m,n>0} (v^m w^{-n} - v^n w^{-m}) q^{mn} \right) \tag{4.16}$$

where[10] $v = e^{2\pi i z}$, $w = e^{-2\pi i \alpha}$, and $q = e^{2\pi i \tau}$.

As noted in [Cha22], we can identify v as a coordinate on a concentric genus one Schottky cover Γ (Figure 6), with $\gamma_1 : p \mapsto qp$. With this identification, we can instead write the Kronecker function on the concentric Schottky cover as

$$F_{\mathrm{Schottky}}(v, w) = \sum_{n \in \mathbb{Z}} \frac{q^n}{q^n v - 1} w^{-n}. \tag{4.17}$$

[9]Different sources cite different identities, e.g. [Mat19] uses a Riemann identity [Mum83]. However, at genus one, these identities are equivalent.

[10]Note that the definition of w differs from what is used in [BL11]. Their identification more uniformly defines exponentiated variables; however, identifying w as done in this thesis makes it more clear as the quasiperiodic factor when z is shifted by a $\mathfrak{B}$-cycle.

verifying the desired properties of the Kronecker function as,

$$\begin{aligned} z \mapsto z+1 &\text{ corresponds to } v \mapsto v \implies F_{\text{Schottky}} \text{ invariant}, \\ z \mapsto z+\tau &\text{ corresponds to } v \mapsto qv \implies F_{\text{Schottky}} \mapsto wF_{\text{Schottky}}, \\ z=0 &\text{ corresponds to } v=1 \text{ where } \operatorname{res}_{v=1} F_{\text{Schottky}}(v,w)=1. \end{aligned} \tag{4.18}$$

Let us show the equivalence of the representations in Equation 4.16 and Equation 4.17. By splitting the sum above, we find

$$F_{\text{Schottky}}(v,w) = \sum_{n>0} \frac{q^n}{q^n v - 1} w^{-n} + \frac{1}{v-1} + \sum_{n>0} \frac{q^{-n}}{q^{-n}v-1} w^n, \tag{4.19}$$

where we can expand the fractions with a geometric series

$$\begin{aligned} &= \left(\sum_{n>0} (-q^n w^{-n}) \sum_{m\geq 0} q^{mn} v^m \right) + \frac{1}{v-1} + \left(\sum_{n>0} w^n \frac{1}{v} \sum_{m\geq 0} q^{mn} v^{-m} \right) \\ &= \left(-\frac{1}{v} \sum_{m,n>0} v^m w^{-n} q^{mn} \right) - \frac{1}{v}\frac{v}{1-v} + \left(\frac{1}{v}\frac{w}{1-w} + \frac{1}{v} \sum_{m,n>0} v^{-m} w^n q^{mn} \right) \\ &= \frac{1}{2\pi i v} F(z,\alpha). \end{aligned} \tag{4.20}$$

This discrepancy, noticed in [Cha22], is due to the consideration that the object that is invariant under a change of representation to the Schottky cover is the one-form corresponding to the Kronecker function. Indeed, we find

$$dv = d(e^{2\pi i z}) = 2\pi i e^{2\pi i z} dz = 2\pi i v dz \implies F_{\text{Schottky}}(v,w)dv = F(z,\alpha)dz. \tag{4.21}$$

Furthermore, one can go beyond previous works by identifying a representation of the Kronecker at genus one on an arbitrary Schottky cover as

$$\begin{aligned} F_{\text{Schottky}}(v,w)dv &= \sum_{n\in\mathbb{Z}} \frac{q^n dv}{q^n v - 1} w^{-n} \\ &= \sum_{n\in\mathbb{Z}} \frac{d(\gamma_1^n v)}{\gamma_1^n v - x} \frac{P_1' - x}{P_1' - \gamma_1^n} w^{-n}, \end{aligned} \tag{4.22}$$

where γ_1 is the generator of the Schottky group, x is the image of the pole in the new coordinates, and P_1 is a fixed point of the generator. This matches the result on the concentric Schottky cover by identifying $\gamma_1 : v \mapsto qv$, $x = 1$, and $P_1 = \infty$. Establishing the equivalence for an arbitrary Schottky cover relies on considering Moebius transformations of the group, as one can check that $v \mapsto Mv$, $x \mapsto Mx$, $\gamma_1 \mapsto M\gamma_1 M^{-1}$ for $M \in \mathrm{PSL}(2,\mathbb{C})$ leaves the expression unchanged.

4.2.3 Eisenstein series representation

The last definition takes advantage of the Eisenstein function E_j and Eisenstein series e_j. These are defined as

$$E_j(z,\tau) = \lim_{L\to\infty} \sum_{m,n=-L}^{L} \frac{1}{(z+m+n\tau)^j} \quad ; \quad e_j(\tau) = \sum_{\substack{m,n\in\mathbb{Z} \\ (m,n)\neq(0,0)}}^{L} \frac{1}{(m+n\tau)^j}, \tag{4.23}$$

where the careful limits on the sum for E_j are necessary to correctly interpret the case when $j = 1$ with slower convergence. In particular, though the sum for E_1 seems to include every combination of (m, n), which would typically make it periodic, one actually finds that

$$E_1(z + \tau, \tau) = E_1(z, \tau) - 2\pi i. \tag{4.24}$$

For $j > 1$, since $E_{j+1}(z, \tau) = -\frac{1}{j}\partial_z E_j(z)$, we find that periodicity is restored.

With these definitions in mind, we find that the Kronecker function is defined as [BL11]

$$F(z, \alpha) = \frac{1}{\alpha} \exp\left(-\sum_{j\geq 1} \frac{(-\alpha)^j}{j}(E_j(z, \tau) - e_j(\tau))\right), \tag{4.25}$$

for which the periodicity is clear from the periodicity of E_j, and the residue can be found through a careful calculation noticing that $E_j(z, \tau) - e_j(\tau) \simeq z^{-j}$ for small z, and using the Taylor series for a logarithm in the exponential.

This representation finds use in research due to the connections the Eisenstein series has to results in number theory, such as revealing relations between multiple zeta values [BMS16]. However, we will not build upon this representation in this thesis, as it is not yet clear how it may be generalized to higher genus. Instead, such questions are left open, and are mentioned in Section 9.1.5.

4.2.4 Periodic analogue

As a consequence of the Liouville theorem, functions on the torus cannot be both periodic and meromorphic. In the construction above, we have defined only holomorphic and meromorphic objects, and so the kernels are quasiperiodic as a result (Equation 4.10). However, in some applications, it is preferable to give up the holomorphicity for complete periodicity of the generating function and integration kernels. This is the approach taken by the definitions in [BL11], the applications in [BDDT18], and recent approaches to generalizations as in [DHS23].

With the quasiperiodicity above, it is easy to modify the Kronecker function by finding a factor $C(z, \alpha)$ satisfying $C(z + \tau, \alpha) = e^{2\pi i\alpha}C(z)$ and $C(0, \alpha) = 1$. A suitable choice is

$$C(z) = \exp\left(2\pi i\alpha \frac{\text{Im}(z)}{\text{Im}(\tau)}\right), \tag{4.26}$$

resulting in the periodic Kronecker function

$$\Omega(z, \alpha) = \exp\left(2\pi i\alpha \frac{\text{Im}(z)}{\text{Im}(\tau)}\right) F(z, \alpha). \tag{4.27}$$

Naturally, we then have

$$\Omega(z + \tau, \alpha) = \Omega(z, \alpha), \tag{4.28}$$

as well as periodicity for the kernels f generated by $\alpha\Omega(z, \alpha) = \sum_{m=0}^{\infty} f^{(m)}(z)\alpha^m$,

$$f(z + \tau) = f(z). \tag{4.29}$$

5 Mathematical conventions for higher genera

5.1 Identification of the pole

At lower genera, when seeking a differential with a pole at x, we would simply write $dz/(z - x)$ or $g^{(1)}(z - x)dz$. This notation is suitable on the corresponding universal covers due to the translation invariance of those surfaces. However, as we go to higher genus, there does not exist a translation invariant cover, so the pole must be explicitly identified separately from the variable. Thus, we will change the labeling of the Kronecker function as

$$F(z-x,\alpha)dz \mapsto F(z,x,\alpha)dz. \tag{5.1}$$

In general, we will reserve the first argument for the parameter corresponding to the differential, the second argument for the parameter corresponding to the location of the pole, and the third argument for the variable related to power-counting or quasiperiodicity.

5.2 Differential structure

At higher genus, it will become ever more important to consider what the Kronecker function looks like in different representations. As we transform between representations, we end up with different charts for our manifold $\mathcal{M}$, which imposes transformation rules on coefficients of differentials

$$\omega = f(z)dz = \tilde{f}(\varphi(z))d(\varphi(z)) \implies \tilde{f}(\varphi(z)) = \varphi'(z)f(z). \tag{5.2}$$

This is also true when considering how our objects transform when a point is moved by a cycle[11], where

$$\omega(z+\mathfrak{B}_j) = \omega(\gamma_j(z)) = f(\gamma_j z)d(\gamma_j z) = \gamma_j'(z)f(\gamma_j z)dz. \tag{5.3}$$

In order to study the objects in different representations, and have better control over their quasiperiodicities it is more appropriate to study the differential forms, which are not affected by the transformation properties. Thus, we will redefine F to be a *Kronecker form*, which will be a generating function for differentials. At genus one, this corresponds to redefining

$$F(z,x,\alpha)\underbrace{dz}_{\text{on universal cover}} = F(z,x,\alpha)\overbrace{\omega(z)}^{\text{in general}} \mapsto F(z,x,\alpha). \tag{5.4}$$

With this differential structure in mind, the unit residue of the Kronecker function corresponds to identifying

$$\mathrm{res}_{z=x}F(z,x,\alpha) = dz, \tag{5.5}$$

which is a convention we will keep as we go to higher genus.

5.3 Quasiperiodicity

At higher genera, we will have a larger homotopy group to consider, which will require keeping track of a greater number of quasiperiodicities. The appropriate generalization of the Kronecker function preserves its invariance under $\mathfrak{A}$-cycles, and multiplicative monodromies under $\mathfrak{B}$-cycles which we will label with variables w_j,

$$F(z+\mathfrak{A}_j,x,\vec{w}) = F(z,x,\vec{w}) \quad ; \quad F(z+\mathfrak{B}_j,x,\vec{w}) = w_j F(z,x,\vec{w}). \tag{5.6}$$

At genus one, recall that $F(z+\tau,\alpha) = \exp(-2\pi i\alpha)F(z,\alpha)$, so we can identify $w_1 = \exp(-2\pi i\alpha)$. As we go to higher genus, depending on the representation, we will make different choices for the meaning of the variables w_j.

[11] The only reason this didn't affect the properties of the Kronecker function at genus one is that on the universal cover, $\gamma(z) = z+\tau \implies \gamma'(z) = 1$.

In Section 7, we will work with commutative w_j, which we can identify with $\exp(-2\pi i\alpha_j)$, since that is the natural quasiperiodicity produced by theta functions. Essentially, the use of the Abel map will restrict us to working with the homology group.

In Section 8, however, we have the freedom to use arbitrary non-commutative w_j, which we will identify with $\exp(b)$, using b as a power-counting variable. In this case, we will be able to manifest quasiperiodic factors corresponding to the full homotopy group.

6 Basis of quasiperiodic differentials

In the study of compact Riemann surfaces, we are familiar with the basis of periodic holomorphic differentials (Equation 2.2). As we seek to understand the Kronecker form, a one-form with a simple pole and power-counting variables expressed in its quasiperiodicity, we realize that this amounts to studying the space of all such quasiperiodic differential forms with simple poles.

In this section, we will demonstrate that the quasiperiodicity of differential forms is closely related to the presence of simple poles in Section 6.2. Then in Section 6.3, we will prove a special case of the Riemann bilinear identity for quasiperiodic holomorphic forms, relating an integral over the two-form $\bar{H} \wedge H$ to the periods H has over the $\mathfrak{A}$ and $\mathfrak{B}$-cycles. Finally, putting these together, we will prove a new result about the dimension of the space of quasiperiodic forms. Holomorphic quasiperiodic forms are $(h-1)$-dimensional space, and meromorphic quasiperiodic forms with a single simple pole are an h dimensional space.

In particular, at genus one, the Kronecker form is the unique quasiperiodic differential form, demonstrating its significance. At higher genus, we have a space of possible Kronecker forms to explore, analogous to the higher-dimensional space of periodic holomorphic forms. The representations of the basis necessary to span this space will be given as Kronecker forms in Section 7 and Section 8.

6.1 Conventions for quasiperiodic differentials

We will use $H(z, \{w_j\}_{j=1}^h)$ to refer to holomorphic differentials, satisfying periodicity properties

$$H(z+\mathfrak{A}_j, \{w_j\}_{j=1}^h) = H(z, \{w_j\}_{j=1}^h) \; \forall j \in \{1, \cdots, h\} \tag{6.1}$$

$$H(z+\mathfrak{B}_j, \{w_j\}_{j=1}^h) = w_j H(z, \{w_j\}_{j=1}^h) \; \forall j \in \{1, \cdots, h\}, \tag{6.2}$$

where w_j is an arbitrary element of some algebra.

We will also use the letters F and G when appropriate, referring to differentials with the same quasiperiodicities; usually using F to refer to differentials with a known and controlled pole, and G to refer to arbtirary meromorphic differentials.

Knowing the quasiperiodicity properties above, we may actually decompose the forms into components. Depending on whether our algebra of quasiperiodicities is commuting, we will have[12]

$$[w_i, w_j] = 0 \ \forall i,j \implies H(z, \{w_j\}_{j=1}^h) = \sum_{\vec{n}\in\mathbb{Z}} \left(\prod_{j=1}^h w_j^{-n_j}\right) h_{\vec{n}}(z), \text{ where } h_{\vec{n}}(z+\mathfrak{B}_j, \cdots) = h_{\vec{n}+\vec{\delta}_j}(z) \tag{6.3}$$

$$[w_i, w_j] \neq 0 \ \forall i,j \implies H(z, \{w_j\}_{j=1}^h) = \sum_{\gamma\in\Gamma} W(\gamma^{-1}) h(\gamma z), \text{ where } W: \gamma_{i_1}^{n_1}\cdots\gamma_{i_s}^{n_s} \mapsto w_{i_1}^{n_1}\cdots w_{i_s}^{n_s} \tag{6.4}$$

where we use the notation from the Schottky group as a convenient shorthand for working with non-commutative $\mathfrak{B}$-cycle transformations, and $(\vec{\delta}_j)_i = \delta_{ij}$.

6.2 Quasiperiodicity and poles

The first step to understanding the space of functions is to recognize how quasiperiodicity makes $\mathfrak{A}$-periods change, producing non-zero integrals along the boundary of the fundamental domain which must be related to poles contained therein.

Consider the integral

$$I = \oint_{\partial\mathcal{L}} H = \sum_{j=1}^h \left(\int_P^{P+\mathfrak{A}_j} + \int_{P+\mathfrak{A}_j}^{P+\mathfrak{A}_j+\mathfrak{B}_j} + \int_{P+\mathfrak{A}_j+\mathfrak{B}_j}^{P+\mathfrak{B}_j} + \int_{P+\mathfrak{B}_j}^{P}\right) H = \sum_{j=1}^h (1-w_j) \int_P^{P+\mathfrak{A}_j} H. \tag{6.5}$$

Its path bounds the fundamental domain on the Schottky cover, at which point we can use the Cauchy residue theorem to identify

$$I = 2\pi i \sum_{x\in\mathcal{L}} \operatorname{res}_{z=x} H(z). \tag{6.6}$$

Thus, when H is holomorphic, we find that $\sum_{j=1}^h (1-w_j) \int_P^{P+\mathfrak{A}_j} H = 0$.

6.3 Riemann bilinear identity for quasiperiodic holomorphic forms

The goal of this section is to show that

$$\oint_{\mathfrak{A}_j} H = 0 \ \forall \ j, \{w_j\} \implies H \equiv 0, \tag{6.7}$$

so the sum of the $\mathfrak{A}$-periods of a holomorphic differential vanishing implies that the differential itself was zero. Our proof will follow a similar structure to [Ber10, Proposition 3.1.2], using a special case of a modified version of the Riemann bilinear identity.

We will study integrals over the two-form $\bar{H} \wedge H$, where $\bar{H}$ is defined as

$$[w_i, w_j] = 0 \ \forall i,j : H = \sum_{\vec{n}\in\mathbb{Z}} \left(\prod_{j=1}^h w_j^{-n_j}\right) h_{\vec{n}}(z), \quad \bar{H} = \sum_{\vec{n}\in\mathbb{Z}} \left(\prod_{j=1}^h w_j^{n_j}\right) \bar{h}_{\vec{n}}(z), \tag{6.8}$$

$$[w_i, w_j] \neq 0 \ \forall i,j : H = \sum_{\gamma\in\Gamma} W(\gamma^{-1}) h(\gamma z), \quad \bar{H} = \sum_{\gamma\in\Gamma} W(\gamma) h(\gamma z), \tag{6.9}$$

[12]Technically, each formula requires an extra w-dependent constant in case our differential includes terms that don't have integer powers of w. For the sake of clarity, this constant is omitted, as it does not affect any of the upcoming proofs, and does not appear in practice in future sections.

such that $\bar{H} \wedge H$ is periodic.

Then, when considering an integral of $\bar{H} \wedge H$ over the simply connected domain $\mathcal{L}$ (Figure 1), we can expand each one-form into components and we find that the w-independent terms correspond to an integral over strictly non-negative terms, equal to 0 only when the entire function H vanishes. For $[w_i, w_j] \neq 0 \; \forall i, j$,

$$\begin{aligned}\int_{\mathcal{L}} (-i)\bar{H} \wedge H &= \int_{\mathcal{L}} (-i) \sum_{\gamma_1 \in \Gamma} \sum_{\gamma_2 \in \Gamma} \bar{h}(\gamma_1 z) h(\gamma_2 z) W(\gamma_1 \gamma_2^{-1}) d\bar{z} \wedge dz \\ &= \int_{\mathcal{L}} \left(\sum_{\gamma \in \Gamma} |h(\gamma z)|^2 + (\text{terms containing } w) \right) dx \wedge dy.\end{aligned} \tag{6.10}$$

with an analogous result for $[w_i, w_j] = 0 \; \forall i, j$. Thus, we see that $\int_{\mathcal{L}} (-i)\bar{H} \wedge H = 0$ for arbitrary w, implies that $H \equiv 0$.

On the other hand, defining a function $\tilde{H}(z) = \int_p^z H$, such that $d(\bar{H}\tilde{H}) = \bar{H} \wedge H$, we find using Stokes theorem

$$\begin{aligned}\int_{\mathcal{L}} (-i)\bar{H} \wedge H &= (-i) \int_{\partial L} \bar{H}\tilde{H} \\ &= (-i) \sum_{j=1}^{h} \left(\int_P^{P+\mathfrak{A}_j} + \int_{P+\mathfrak{A}_j}^{P+\mathfrak{A}_j+\mathfrak{B}_j} + \int_{P+\mathfrak{A}_j+\mathfrak{B}_j}^{P+\mathfrak{B}_j} + \int_{P+\mathfrak{B}_j}^{P} \right) \bar{H}\tilde{H} \\ &= (-i) \sum_{j=1}^{h} \left[\left(\int_P^{P+\mathfrak{A}_j} \bar{H}\tilde{H} - \int_{P+\mathfrak{B}_j}^{P+\mathfrak{A}_j+\mathfrak{B}_j} \bar{H}\tilde{H} \right) + \left(\int_{P+\mathfrak{A}_j}^{P+\mathfrak{A}_j+\mathfrak{B}_j} \bar{H}\tilde{H} - \int_P^{P+\mathfrak{B}_j} \bar{H}\tilde{H} \right) \right] \\ &= (-i) \sum_{j=1}^{h} \left[\int_P^{P+\mathfrak{A}_j} \bar{H}(\tilde{H}(z) - \tilde{H}(z+\mathfrak{B}_j)) + \int_P^{P+\mathfrak{B}_j} \bar{H}(\tilde{H}(z+\mathfrak{A}_j) - \tilde{H}(z)) \right] \\ &= i \sum_{j=1}^{h} \left[\oint_{\mathfrak{B}_j} \bar{H} \oint_{\mathfrak{A}_j} H - \oint_{\mathfrak{A}_j} \bar{H} \oint_{\mathfrak{B}_j} H \right].\end{aligned} \tag{6.11}$$

Consequently $\oint_{\mathfrak{A}_j} H = 0 \; \forall \; j, \{w_j\} \implies \int_{\mathcal{L}} (-i)\bar{H} \wedge H = 0 \; \forall \; \{w_j\} \implies H \equiv 0$. In particular, this means that the dimension of the space of quasiperiodic holomorphic forms does not exceed $h - 1$, which we can prove by contradiction.

Suppose we are be able to find h linearly independent H_i. Considering the $h \times h$ matrix $A_{ij} = \oint_{\mathfrak{A}_i} H_j$, we know it cannot have a rank greater than $h - 1$ since as a result of Section 6.2, we have that $\mathfrak{A}$ periods for a holomorphic form can form a linear combination that vanishes. Since the matrix is not of full rank, there must also be a linear combination of H_j that vanishes, our assumption that h linearly independent H_i existed was false.

6.4 Construction of an arbitrary form given a basis

Having understood that the space of quasiperiodic differential forms is of at most dimension $h - 1$, let us show how we could construct arbitrary functions given a basis with that many independent elements. Explicitly showing an existence of such a basis is left for later sections (Section 7 and Section 8).

Using the result of Section 6.2, we know that the $\mathfrak{A}$-periods of a holomorphic differential form satisfy a linear relation. This means that only $h - 1$ of the $\mathfrak{A}$-periods are independent, so we will choose to ignore $\mathfrak{A}_h$. Suppose we have $h-1$ holomorphic forms H_i that are independent; in particular, we find that the matrix $A_{ij} = \oint_{\mathfrak{A}_i} H_j$ is invertible. Then, for some unknown H, we can identify the vector of $\mathfrak{A}$ periods $P_i = \oint_{\mathfrak{A}_i} H$. Then, writing $c_i = (A^{-1})_{ij} P_j$ we find that

$$\oint_{\mathfrak{A}_k} H - \left(\sum_{j=1}^{h} c_j H_j \right) = 0 \ \forall k = 1, \cdots, h, \tag{6.12}$$

and so by the conclusion of Section 6.3, since all $\mathfrak{A}$-periods vanish, including $\mathfrak{A}_h$, we must have that the differential $H - \sum_{j=1}^{h} c_j H_j$ vanishes too. Thus, $H = \sum_{j=1}^{h} c_j H_j$.

To find a basis for general quasiperiodic meromorphic forms, we must include an extra element in our basis that may contribute to the pole. Suppose we have quasiperiodic form $F(z,x)$ with a single simple pole satisfying $\text{res}_{z=x} F(z,x) = dz$, for arbitrary $x \in \mathcal{L}$. Then, we may write an quasiperiodic meromorphic form G with simple poles with residue $r_i dz$ at points x_i as

$$G = \sum_i r_i F(z, x_i) + H; \quad H \text{ holomorphic}, \tag{6.13}$$

where we use the result above for holomorphic forms to express H in the basis. If we fix the location of the simple pole to some x, then we find that the possible functions are spanned by $\{F(z,x), H_1, \cdots, H_{h-1}\}$, so the space of quasiperiodic meromorphic forms with a single simple pole is precisely h-dimensional, analogous to the space of periodic holomorphic forms.

In future sections, we will find bases that look like $\{F_j\}_{j=1}^{h}$, since the controlled unit residue of such functions naturally corresponds to a generalization of the Kronecker function. In order to recover a basis with $h-1$ holomorphic functions, one may write

$$H_j = F_{j+1} - F_j \ \forall \ j = 1, \cdots, h-1. \tag{6.14}$$

The connection between holomorphic forms and quasiperiodic forms will be most clear in Section 8, as each element of the basis F_j will correspond precisely to ω_j.

7 Kronecker forms as ratios of theta functions

7.1 Definition

With the changes done to the conventions made above in Section 5, we should rewrite the genus one Kronecker form from Equation 4.13 as

$$F(z, x, \alpha) = \frac{\theta(z - x + \alpha)}{\theta(z-x)\theta(\alpha)} d_z(\theta(z-p))\bigg|_{p=z}, \tag{7.1}$$

where z and x are identified on the universal cover, and θ is the unique odd theta function at genus one. As we generalize to higher genus, the θ function depends on vectors in $\mathbb{C}^h$, which most naturally corresponds to using the image of z and x under the Abel map $\mathfrak{u}$. With this in mind, we consider the generalization

$$F_D(z, x, \vec{\alpha}) = \frac{\theta_D(\mathfrak{u}(z) - \mathfrak{u}(x) + \vec{\alpha})}{\theta_D(\mathfrak{u}(z) - \mathfrak{u}(x))\theta_D(\vec{\alpha})} d_z(\theta_D(\mathfrak{u}(z) - \mathfrak{u}(p)))\bigg|_{p=z}, \tag{7.2}$$

where D is a divisor that includes the basepoint of the Abel map (Section 2.4), $z, x \in \mathcal{M}$ are points on the manifold, and $\vec{\alpha} \in \mathbb{C}^h$ will serve as powercounting variables[13]. We will explore

[13]Technically, one may choose to keep α_j as arbitrary commuting variables. This keeps most of the properties, in particular the residue and quasiperiodicity that we desire from Kronecker form. However, we will find that the Fay identity relies on having these vectors live in $\mathbb{C}^h$.

the properties of this Kronecker form, several of which appear only when the divisor is chosen such that the theta function is odd. As pointed out in Equation 2.14, we will denote the theta function as θ when the theta function is odd, where we make an arbitrary, but consistent, choice of characteristics. Similarly, we will denote the corresponding odd Kronecker form by F.

In the upcoming sections, we will use F_D when the result applies for any divisor that includes the basepoint, and F when the result applies only to odd Kronecker forms.

7.2 Properties

7.2.1 Residue and periodicity

When $z = x$ the theta function in the denominator vanishes[14], so we reproduce

$$\mathrm{res}_{z=x} F_D(z, x, \vec{\alpha}) = \lim_{z\to x} \frac{\theta_D(\mathfrak{u}(z) - \mathfrak{u}(x) + \vec{\alpha})}{(z - x)\partial_z(\theta_D(\mathfrak{u}(z) - \mathfrak{u}(x)))\theta_D(\vec{\alpha})} d_z(\theta_D(\mathfrak{u}(z) - \mathfrak{u}(p)))\Big|_{p=z} = dz. \quad (7.3)$$

We also reproduce the quasiperiodicity properties,

$$\begin{aligned} F_D(z + \mathfrak{B}_j, x, \vec{\alpha}) &= \frac{\exp(2\pi i(\mathfrak{u}_j(z - x) - \alpha_j))\theta_D(\mathfrak{u}(z) - \mathfrak{u}(x) + \vec{\alpha})}{\exp(2\pi i \mathfrak{u}_j(z - x))\theta_D(\mathfrak{u}(z) - \mathfrak{u}(x))\theta_D(\vec{\alpha})} d_z(\theta_D(\mathfrak{u}(z) - \mathfrak{u}(p)))\Big|_{p=z} \\ &= \exp(-2\pi i\alpha_j)F_D(z, x, \vec{\alpha}) = F_D(z, x - \mathfrak{B}_j, \vec{\alpha}), \end{aligned} \quad (7.4)$$

where we can identify $w_j = \exp(-2\pi i\alpha_j)$.

7.2.2 Fay identity

When working with odd theta functions, we can then use the the Fay trisecant identity [Mum84]. For points a, b, c, d on a genus h compact Riemann surface, and a vector $\vec{z} \in \mathbb{C}^h$ it reads

$$\begin{aligned} &\Theta(\vec{z} + \mathfrak{u}(c) - \mathfrak{u}(a))\Theta(\vec{z} + \mathfrak{u}(d) - \mathfrak{u}(b))\theta(\mathfrak{u}(c) - \mathfrak{u}(b))\theta(\mathfrak{u}(a) - \mathfrak{u}(d)) \\ +&\Theta(\vec{z} + \mathfrak{u}(c) - \mathfrak{u}(b))\Theta(\vec{z} + \mathfrak{u}(d) - \mathfrak{u}(a))\theta(\mathfrak{u}(c) - \mathfrak{u}(a))\theta(\mathfrak{u}(d) - \mathfrak{u}(b)) \\ =&\Theta(\vec{z} + \mathfrak{u}(c) + \mathfrak{u}(d) - \mathfrak{u}(a) - \mathfrak{u}(b))\Theta(\vec{z})\theta(\mathfrak{u}(c) - \mathfrak{u}(d))\theta(\mathfrak{u}(a) - \mathfrak{u}(b)), \end{aligned} \quad (7.5)$$

A calculation, done in Appendix B, reveals that this identity can be transformed into one that applies to odd Kronecker forms

$$F(b, a, \vec{\alpha})F(d, a, \vec{\beta}) = F(b, d, \vec{\alpha})F(d, a, \vec{\alpha} + \vec{\beta}) + F(b, a, \vec{\alpha} + \vec{\beta})F(d, b, \vec{\beta}) \quad (7.6)$$

provided that $\vec{\alpha} \in \mathbb{C}^h$ and $\vec{\beta} = \mathfrak{u}(c) - \mathfrak{u}(d)$ for some $c \in \mathcal{M}$.

7.2.3 Expression as ratio of prime forms

If we choose $\vec{\alpha} = \mathfrak{u}(y) - \mathfrak{u}(z)$ for some $y \in \mathcal{M}$, one notices that

$$F_E(z, x, y) = F(z, x, \mathfrak{u}(y) - \mathfrak{u}(z)) = \frac{E(y, x)}{E(z, x)E(y, z)}, \quad (7.7)$$

[14] At a glance, it may seem that we have residues at other points since our divisor includes several points, e.g. by having $\mathfrak{u}(x) = 0$ and $z \in D$. However, a careful calculation reveals that the differential $d_z(\theta_D(\mathfrak{u}(z) - \mathfrak{u}(p)))\Big|_{p=z}$ vanishes for $z \in D$ with $\mathfrak{u}(z) \neq 0$, cancelling the divergent contribution.

independently of the exact characteristics used for the odd theta functions. Essentially, one might say that the odd Kronecker forms, as functions of $\vec{\alpha}$, agree on a particular submanifold.

However, this identification produces several challenges that restrict its further use. Identifying $\vec{\alpha} = \mathfrak{u}(y) - \mathfrak{u}(z)$ makes the quasiperiodic factors $w_j = \exp(-2\pi i \alpha_j)$ dependent on z, reducing our control over the quasiperiodic behavior. It also restricts our $\vec{\alpha}$ to a particular z-dependent one-dimensional submanifold, which prevents a full expansion as a power series of α_j.

7.2.4 Kronecker forms as basis

One expects that Kronecker forms defined using theta functions of different characteristics are linearly independent. This means that the results of Section 6 should apply, noting that for theta functions we must have commuting quasiperiodicities, as theta functions do not work with non-commuting variables.

We can verify that the Kronecker form can be expressed as a series in w_j, as in Equation 6.3, by identifying $w_j = \exp(-2\pi i \alpha_j)$, and expanding the theta functions containing $\vec{\alpha}$.

$$\begin{aligned}\theta\left[\begin{smallmatrix}\epsilon\\ \epsilon'\end{smallmatrix}\right](\vec{v}+\vec{\alpha}) &= \sum_{\vec{n}\in\mathbb{Z}^h} \exp(2\pi i[(n_j+\epsilon_j)\tau_{jk}(n_k+\epsilon_k) + (n_j+\epsilon_j)(v_j+\alpha_j+\epsilon'_j)]) \\ &= \sum_{\vec{n}\in\mathbb{Z}^h} \left(\prod_{i=1}^{h} w_i^{-n_i-\epsilon_i}\right) \exp(2\pi i[(n_j+\epsilon_j)\tau_{jk}(n_k+\epsilon_k) + (n_j+\epsilon_j)(v_j+\epsilon'_j)]),\end{aligned} \tag{7.8}$$

The terms containing $w_j^{-\epsilon_j}$ will cancel between the numerator and denominator of the Kronecker form, leaving a formula that in principle can give the contributions from each fundamental domain similarly to Equation 6.3.

It is easy to show numerically that we can choose several theta functions with different odd characteristics $\{\theta_i\}_{i=1}^h$, and use the corresponding odd Kronecker forms $\{F_i\}_{i=1}^h$ defined as ratios of these theta functions, and then express arbitrary quasiperiodic differentials, such as other Kronecker forms, in this basis

$$F_D(z, x, \vec{\alpha}) = \sum_{i=1}^{h} c_i(x, \vec{\alpha}) F_i(z, x, \vec{\alpha}). \tag{7.9}$$

Unfortunately, the analytical form for these coefficients is not known, and their dependence on x and $\vec{\alpha}$ makes them challenging to work with. One may hope that, if these coefficients are better understood, that the Fay identities known for odd Kronecker forms may be generalized to other quasiperiodic forms by expressing them in this basis.

7.2.5 Periodic analogue

Recall that at genus one, we had the periodic analogue of the Kronecker function (Equation 4.27),

$$\Omega(z, \alpha) = \exp\left(2\pi i \alpha \frac{\mathrm{Im}(z)}{\mathrm{Im}(\tau)}\right) F_D(z, \alpha). \tag{7.10}$$

The natural generalization of this uses the exponential factor, modifying the expression to use the vectors $\mathfrak{u}(z)$ and $\vec{\alpha}$, as well as using the inverse of the imaginary part, writing

$$\Omega(z, x, \vec{\alpha}) = \exp\left(2\pi i \sum_{i,j} Y_{ij} \mathrm{Im}(\mathfrak{u}_i(z)) \alpha_j\right) F_D(z, x, \vec{\alpha}), \tag{7.11}$$

where $Y = [\mathrm{Im}(\tau)]^{-1}$ so that

$$\sum_i Y_{ij}\mathrm{Im}(\mathfrak{u}_i(z+\mathfrak{B}_k)) = \sum_i Y_{ij}\mathrm{Im}(\mathfrak{u}_i(z)) + \sum_i [\mathrm{Im}(\tau)]^{-1}_{ij}\mathrm{Im}[\tau]_{ik} = \delta_{jk} \tag{7.12}$$

which allows the exponential in Equation 7.11 to pull out the $w_j^{-1} = \exp(2\pi i\alpha_j)$ factor necessary to restore periodicity.

This periodic analogue precisely matches the periodic analogue at genus one from Equation 4.27 and has similarities to other periodic approaches at higher genus [DHS23]. However, it is not explored further in this thesis.

7.2.6 Cyclic products

In [Tsu23, Equation 2.13], Tsuchiya suggests a very similar generalization of the Kronecker function as

$$F_{\mathrm{Tsuchiya}}(z,x,\vec{\alpha}) = \frac{\theta(\mathfrak{u}(z-x)+\vec{\alpha})}{E(z,x)\theta(\vec{\alpha})} \tag{7.13}$$

albeit his prescription for the characteristics is odd only at genera of the form $h = 4n+1, 4n+2$.

However, he uses this Kronecker function only as part of a cyclic product with $z_{N+1} = z_1$

$$\prod_{i=1}^{N} F_{\mathrm{Tsuchiya}}(z_i, z_{i+1}, \vec{\alpha}), \tag{7.14}$$

where if we expand the definition of the prime form (Equation 2.15),

$$\prod_{i=1}^{N} \frac{\theta(\mathfrak{u}(z_i - z_{i+1})+\vec{\alpha})}{\theta(\mathfrak{u}(z_i - z_{i+1}))\theta(\vec{\alpha})}\psi(z_i)\psi(z_{i+1}) = \prod_{i=1}^{N} \frac{\theta(\mathfrak{u}(z_i - z_{i+1})+\vec{\alpha})}{\theta(\mathfrak{u}(z_i - z_{i+1}))\theta(\vec{\alpha})}\psi(z_i)^2 = \prod_{i=1}^{N} F(z_i, z_{i+1}, \vec{\alpha}), \tag{7.15}$$

we recover the generalization presented here.

7.3 Expansion into kernels

7.3.1 Pole-cancellation

Now, we would like to reproduce an expansion of the Kronecker function into integration kernels as was done in Equation 4.4. In order to do this, we need to introduce some pole-cancelling function $P(\vec{\alpha})$ to cancel the Kronecker function's divergence when $\vec{\alpha} = \vec{0}$ such that

$$P(\vec{\alpha})F(z,x,\vec{\alpha}) = \sum_{\vec{n}\in\mathbb{Z}^h_{\geq 0}} g^{(\vec{n})}(z,x)\prod_{j=1}^{h}\alpha_j^{n_j}. \tag{7.16}$$

The most obvious pole-cancellation functions mimic the choice $P(\alpha) = \alpha$ at genus one. For instance, one might choose a product-based pole-cancellation $P(\vec{\alpha}) = \prod_{j=1}^h \alpha_j$ or sum-based pole-cancellation $P(\vec{\alpha}) = \sum_{j=1}^h \alpha_j$. However, as we had seen in Section 2.4, the divergence of $1/\theta(\vec{\alpha})$ extends to a $(h-1)$-dimensional variety in $\mathbb{C}^h$, so in order to have a consistent expansion in components of $\vec{\alpha}$, we need to cancel the divergence on the entire variety[15], such as by using

[15] Some caution is to be taken with restricting the pole-cancellation functions in this way. It is possible that we must instead restrict $\vec{\alpha}$ to a submanifold in which the divergence is simpler (e.g. $\vec{\alpha} \in \{\mathfrak{u}(y) - \mathfrak{u}(z) : y \in \mathcal{M}\}$). However, isolating powers in the Fay identity later in this section requires that these expansions are valid for $\vec{\alpha}$ on a larger space.

$P(\vec{\alpha}) = \theta(\vec{\alpha})$ directly. This choice in particular gives a very convenient expression for the integration kernels, since writing

$$\theta(\vec{\alpha})F(z,x,\vec{\alpha}) = \sum_{\vec{n}\in\mathbb{Z}^h_{\geq 0}} g^{(\vec{n})}(z,x)\prod_{j=1}^{h}\alpha_j^{n_j} \implies g^{(\vec{n})}(z,x) = \frac{\theta^{(\vec{n})}(\mathfrak{u}(z)-\mathfrak{u}(x))}{\theta(\mathfrak{u}(z)-\mathfrak{u}(x))}d_z(\theta(\mathfrak{u}(z)-\mathfrak{u}(p)))\Big|_{p=z} \tag{7.17}$$

The space of possible pole-cancellation functions is explored in Appendix C; however, for the remainder of the thesis we will stick to using an arbitrary odd pole-cancellation function

$$P(\vec{\alpha}) = \sum_{\substack{\vec{n}\,\in\,\mathbb{Z}^h_{\geq 0}\\ \sum_{j=1}^h n_j \text{ odd}}} P^{(\vec{n})}\prod_{j=1}^{h}\alpha_j^{n_j}, \quad P(\vec{\alpha}) = -P(-\vec{\alpha}), \tag{7.18}$$

where the components $P^{(\vec{n})}$ will be used when P itself is expanded in components of $\vec{\alpha}$.

7.3.2 Properties of integration kernels

As in the case at genus one (Equation 4.10), we can use the properties of the Kronecker form to understand the quasiperiodicity and residue of the integration kernels.

$$\begin{gathered} F(z+\mathfrak{B}_j,x,\vec{\alpha}) = \exp(-2\pi i\alpha_j)F(z,x,\vec{\alpha}) \implies g^{(\vec{n})}(z+\mathfrak{B}_j,x) = \sum_{k=0}^{n_j}(-2\pi i)^k g^{(\vec{n}-k\vec{\delta}_j)}(z,x),\\ \mathrm{res}_{z=x}P(\vec{\alpha})F(z,x,\vec{\alpha}) = P(\vec{\alpha})dz \implies \mathrm{res}_{z=x}g^{(\vec{n})}(z,x) = P^{(\vec{n})}dz, \end{gathered} \tag{7.19}$$

where $P^{(\vec{n})}$ is the coefficient of $\prod_{j=1}^h \alpha_j^{n_j}$ in the expansion of P (Equation 7.18). At genus one, when $P(\alpha)=\alpha$, we reproduce that $\mathrm{res}_{z=x}g^{(n)}(z,x) = \delta_{n1}dz$.

7.3.3 Fay identity for integration kernels

With this pole-cancellation in mind, we can proceed to analyze the integration kernels. The most interesting step to take is to attempt to reproduce the procedure by which the Fay identity for the Kronecker function is expanded to become a Fay identity for the integration kernels, reaching a result like Equation 4.12.

We can start by multiplying the Fay identity for the Kronecker function (Equation 7.6) by the corresponding pole-cancellation factors

$$\begin{aligned} P(\vec{\alpha}+\vec{\beta})[P(\vec{\alpha})F(b,a,\vec{\alpha})][P(\vec{\beta})F(c,a,\vec{\beta})] =& P(\vec{\beta})[P(\vec{\alpha})F(b,c,\vec{\alpha})][P(\vec{\alpha}+\vec{\beta})F(c,a,\vec{\alpha}+\vec{\beta})]\\ &+ P(\vec{\alpha})[P(\vec{\alpha}+\vec{\beta})F(b,a,\vec{\alpha}+\vec{\beta})][P(\vec{\beta})F(d,b,\vec{\beta})], \end{aligned} \tag{7.20}$$

so we can now expand all terms in powers of α_j and β_j, and group them by powers

$$\sum_{\vec{m},\vec{n}\in\mathbb{Z}^h_{\geq 0}} \tilde{F}^{(\vec{m};\vec{n})}(a,b,c)\left(\prod_{j=1}^{h}\alpha_j^{m_j}\beta_j^{n_j}\right) = 0, \tag{7.21}$$

where we keep the coefficients of P, products of kernels g, and combinatoric factors in the term

$$\tilde{F}^{(\vec{m};\vec{n})}(a,b,c) = \sum_{\substack{\vec{r}\le\vec{m}\\ \vec{s}\le\vec{n}}} \left[\prod_j \binom{m_j+n_j-r_j-s_j}{m_j-r_j}\right] \begin{bmatrix} g^{(\vec{r})}(b,a)g^{(\vec{s})}(c,a)P^{(\vec{m}+\vec{n}-\vec{r}-\vec{s})} \\ -g^{(\vec{r})}(b,c)P^{(\vec{s})}g^{(\vec{m}+\vec{n}-\vec{r}-\vec{s})}(c,a) \\ -P^{(\vec{r})}g^{(\vec{s})}(c,b)g^{(\vec{m}+\vec{n}-\vec{r}-\vec{s})}(b,a) \end{bmatrix}.$$

At this point, the next step at genus one would be to remove the dependence on both α and β, using derivatives on each variable, and then setting $\alpha = \beta = 0$, giving us simply

$$(\partial_\alpha)^M(\partial_\beta)^N \sum_{m,n\in\mathbb{Z}^h_{\ge 0}} \tilde{F}^{(m;n)}(a,b,c)\,(\alpha^m\beta^n) = 0 \overset{\alpha=\beta=0}{\Longrightarrow} \tilde{F}^{(M;N)}(a,b,c) = 0 \text{ for } h=1, \tag{7.23}$$

which, after setting $P(\alpha) = \alpha$, would give us precisely the identity in Equation 4.12. Isolating the terms corresponding to $\alpha^M\beta^N$ in this way was only possible because the Fay identity at genus one is valid for all $\alpha, \beta \in \mathbb{C}$. However in general we require $\vec{\beta} \in \{\mathfrak{u}(p) - \mathfrak{u}(c) | p \in \mathcal{M}\}$, which restricts our derivatives to be in a particular one-dimensional submanifold parametrized by d.

Instead, we can only take derivatives with respect to the point p, so we would have

$$\begin{aligned} &\prod_{j=1}^h (\partial_\alpha)^{M_j} \left(\frac{d}{dp}\right)^N \sum_{m,n\in\mathbb{Z}^h_{\ge 0}} \tilde{F}^{(m;n)}(a,b,c) \left(\prod_{j=1}^h \alpha_j^{m_j}\beta_j^{n_j}\right) = 0 \\ &\overset{\alpha=0}{\Longrightarrow} \left(\frac{d}{dp}\right)^N \sum_{\vec{n}\in\mathbb{Z}^h_{\ge 0}} \tilde{F}^{(\vec{M};\vec{n})}(a,b,c) \left(\prod_{j=1}^h \beta_j^{n_j}\right) = 0, \end{aligned} \tag{7.24}$$

where we can write

$$\frac{d}{dp} = \partial_p + \sum_{j=1}^h \frac{\partial \mathfrak{u}_j(p)}{\partial p}\partial_{\mathfrak{u}_j(p)} = \partial_p + \sum_{j=1}^h \omega_j(p)\partial_{\mathfrak{u}_j(p)}. \tag{7.25}$$

Thus, it is clear that we can only control the sum of weights N, rather than individual weights $\vec{n}$. Furthermore, the partial derivative ∂_p can act upon $\omega_j(p)$ when $N > 1$, so we will have contributions for all terms where $\sum_{j=1}^h n_j \le N$; terms with a higher sum will vanish by setting $p = c \iff \beta = 0$. Combining all the combinatoric coefficients resulting from the derivatives, for which the full calculation is present in Appendix D, we find the unpleasant result

$$\sum_{\substack{\vec{n},\,\vec{d}\,\in\,\mathbb{Z}^h_{\ge 0}\\ \sum_i n_i \le N\\ \sum_i d_i = N}} \left(\prod_{j=1}^h \sum_{\substack{\vec{p}^j\,\in\,\mathbb{Z}^{d_j}_{\ge 0}\\ \sum_k p^j_k = n_j\\ \sum_k k p^j_k = d_j\\ p^j_0 = 0}} \frac{(N!)(n_j!)}{\prod_{k=1}^{d_j}(k!)^{p^j_k}(p^j_k!)}\left[\omega_j(c)^{p^j_1}\omega'_j(c)^{p^j_2}\cdots\right]\right) \times \tilde{F}^{(\vec{m};\vec{n})}(a,b,c) = 0, \tag{7.26}$$

for all $\vec{m} \in \mathbb{Z}^h_{\ge 0}, N \in \mathbb{Z}_{\ge 0}$ where

$$\tilde{F}^{(\vec{m};\vec{n})}(a,b,c) = \sum_{\substack{\vec{r}\le\vec{m}\\ \vec{s}\le\vec{n}}} \left[\prod_j \binom{m_j+n_j-r_j-s_j}{m_j-r_j}\right] \begin{bmatrix} g^{(\vec{r})}(b,a)g^{(\vec{s})}(c,a)P^{(\vec{m}+\vec{n}-\vec{r}-\vec{s})} \\ -g^{(\vec{r})}(b,c)P^{(\vec{s})}g^{(\vec{m}+\vec{n}-\vec{r}-\vec{s})}(c,a) \\ -P^{(\vec{r})}g^{(\vec{s})}(c,b)g^{(\vec{m}+\vec{n}-\vec{r}-\vec{s})}(b,a) \end{bmatrix}, \tag{7.27}$$

where ω_j and ω'_j are the components and derivatives of the components of the holomorphic differentials in some local chart.

8 Kronecker forms on Schottky covers

8.1 Definition

In Equation 4.22, we found a representation of the Kronecker function at genus one as a sum over the group. With the conventions from Section 5, we can write the genus one Kronecker form as[16]

$$F(z,x,w_1|\Gamma) = \sum_{\gamma\in\Gamma} \frac{d(\gamma z)}{\gamma z - x}\frac{P_1 - x}{P_1 - \gamma z} W(\gamma^{-1}) = \sum_{\gamma\in\Gamma} \frac{dz}{z-\gamma x}\frac{\gamma P_1 - \gamma x}{\gamma P_1 - z} W(\gamma) \tag{8.1}$$

where $W : \gamma_1^n \mapsto w_1^n$ keeps track of the monodromies, and the two expressions are related to each other by relabeling $\gamma \mapsto \gamma^{-1}$ and applying a Moebius transformation to each term to move the element's action to the other variables. Often, the second form will be more convenient since it does not include any elements of the Schottky group acting within our differential.

Generalizing this function to a Kronecker form is straightforward, where we can write

$$F_j(z,x,\vec{w}|\Gamma) = \sum_{\gamma\in\Gamma} \frac{dz}{z-\gamma x}\frac{\gamma P_j - \gamma x}{\gamma P_j - z} W(\gamma), \tag{8.2}$$

where $W : \gamma_{i_1}^{n_1}\cdots\gamma_{i_s}^{n_s} \mapsto w_{i_1}^{n_1}\cdots w_{i_s}^{n_s}$ generalizes how the monodromies are kept track of with the non-commutative w_j variables. There is a choice to make on which fixed point is chosen in the Moebius invariant term. As we will see in later sections, the Kronecker forms corresponding to different choices of the fixed point are independent, and form a basis satisfying the requirements given in Section 6.

This expression is very similar to the definition of the holomorphic differential forms from Equation 3.6, with each term in the sum being Moebius invariant. However, instead of using both fixed points we control the location of a pole using the variable x, and include the function $W(\gamma)$ to keep track of power-counting variables.

8.2 Properties

8.2.1 Residue and quasiperiodicity

It is easy to verify the properties we desire from a Kronecker form,

$$\begin{aligned} F_j(\gamma_k z, x, \vec{w}|\Gamma) &= w_k F_j(z,x,\vec{w}|\Gamma), \\ \mathrm{res}_{z=x} F_j(\gamma_k z, x, \vec{w}|\Gamma) &= dz, \end{aligned} \tag{8.3}$$

where the first comes from relabeling the sum to aborb the γ_k into the generator, and the second simply isolates the residue from the $\gamma = \mathrm{id}$ term.

Unlike the representation with the theta Kronecker forms, where we had $F(z+\mathfrak{B}_j, x, \vec{\alpha}) = F(z, x-\mathfrak{B}_j, \vec{\alpha})$, the Schottky Kronecker forms do not have a simple symmetry with regards to quasiperiodicity in their auxiliary variable. A calculation done in Appendix E reveals that we instead have

$$F_j(z,\gamma_k x,\vec{w}|\Gamma) = F_j(z,x,\vec{w}|\Gamma) + F_k(z,x,\vec{w}|\Gamma)(w_k^{-1} - 1). \tag{8.4}$$

[16]Note that we have changed convention from using P_1' to using P_1. At genus one, the choice to use P_1' is often simpler, since we identify $\gamma_1 : z \mapsto qz$ with $|q| = |\exp(2\pi i\tau)| < 1$, but we would like our fixed point to be at infinity. As a consequence of the genus one Kronecker form being the unique quasiperiodic differential, it turns out that the choice did not actually matter, and so we will stick with P_1 to simplify notation and match the higher genus conventions.

8.2.2 Basis of quasiperiodic differentials

By carefully studying the residues at the poles in a disc, we find

$$\oint_{\mathfrak{A}_i} F_k(z,x,\vec{w}|\Gamma) = -2\pi i \delta_{ik} \sum_{n=0}^{\infty} W(\gamma_k^{-n}) = 2\pi i \frac{\delta_{ik}}{w_k^{-1} - 1}, \tag{8.5}$$

which clearly demonstrate the independence of the Kronecker forms F_j. Furthermore, they are normalized such that the periods $\oint_{\mathfrak{A}_i} F_k$ vanish if $i \neq k$. One may choose to include the remaining factors could in the definition of the Kronecker form, normalizing them such that only the δ_{ik} remains, just like the basis of periodic holomorphic forms (Equation 2.2).

8.2.3 Sum as average of lower genus objects

One way to approach evaluation of the Schottky Kronecker form is to identify lower genus objects in the sum representation. Splitting the fraction in the expression, we find

$$F_j(z,x,\vec{w}|\Gamma) = \sum_{\gamma\in\Gamma} \left[\frac{dz}{z-\gamma x} - \frac{dz}{z-\gamma P_j} \right] W(\gamma), \tag{8.6}$$

where we are essentially summing over the genus zero integration kernels. At this point, one could in principle directly integrate, finding a generating function for length 1 polylogarithms

$$E(z,x,\vec{w}|\Gamma) = \int_{z_0}^{z} F_j(z,x,\vec{w}|\Gamma) = \sum_{\gamma\in\Gamma} \log\left[\frac{(z-\gamma x)(z_0 - \gamma P_j)}{(z-\gamma P_j)(z_0 - x)} \right] W(\gamma), \tag{8.7}$$

generalizing the statement given in [BL11, Section 6.1]. Essentially, using the Schottky cover, we can see how Kronecker forms at arbitrary genus are simply averages over genus zero integration kernels, labeled with words corresponding to the elements of the homotopy group.

However, expanding directly at the lowest genus misses the fact that we know the Kronecker form should have a pole in the power-counting variable, as $F(z,\alpha) \sim 1/\alpha$ at genus one. This uncaptured pole leads to difficulties if one tries to directly use the expansion at genus zero. It is more useful then to try to find a way to reduce the Kronecker form to the well-understood expressions at genus one. This can be done by splitting elements $\gamma \in \Gamma$ into $\gamma = \gamma_j^n \tilde{\gamma}$ for $\tilde{\gamma} \in \Gamma_j \backslash \Gamma$ and $n \in \mathbb{Z}$,

$$\begin{aligned} F_j(z,x,\vec{w}|\Gamma) &= \sum_{\tilde{\gamma}\in\Gamma_j\backslash\Gamma} \sum_{n\in\mathbb{Z}} W(\tilde{\gamma}^{-1}\gamma_j^{-n}) \frac{d(\gamma_j^n \tilde{\gamma} z)}{\gamma_j^n \tilde{\gamma} z - x} \frac{P_1 - x}{P_1 - \gamma_j^n \tilde{\gamma} z} && (8.8) \\ &= \sum_{\tilde{\gamma}\in\Gamma_j\backslash\Gamma} W(\tilde{\gamma}^{-1}) F(\tilde{\gamma} z, x, w_j|\Gamma_j), && (8.9) \end{aligned}$$

where $F(z,x,w_j|\Gamma_j)$ is simply the genus one Kronecker form defined on the cover generated by γ_j. Analogously to identifying the Kronecker form as a weighted average of genus zero integration kernels, it is also an average of genus one Kronecker forms. Since we know how to isolate the pole in the power-counting variable at genus one, this expression will be very useful as we begin expanding in Section 8.3.

8.2.4 Degeneration limits

One of the procedures we would like to understand for polylogarithms at higher genus are their limits as the underlying surface degenerates. For example, there is a description of how shrinking an $\mathfrak{A}$-cycle on a genus one surface allows one to relate string amplitudes between a torus and a sphere with two punctures, as one may imagine from the geometric interpretation depicted in Figure 7.

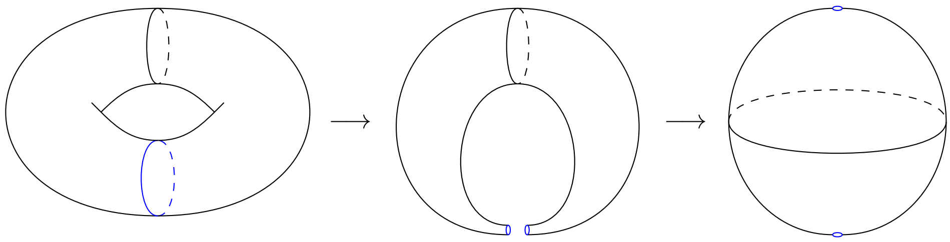

Figure 7: A sketch of the procedure, starting with a torus (left), shrinking and cutting an $\mathfrak{A}$-cycle (middle), and recognizing a sphere with two punctures (right). The image is inspired by Figure 3 from [BK21].

We may imagine a similar procedure done on a Schottky cover, as we shrink the circles corresponding to some $\mathfrak{A}$-cycle down to their fixed point, as in Figure 8.

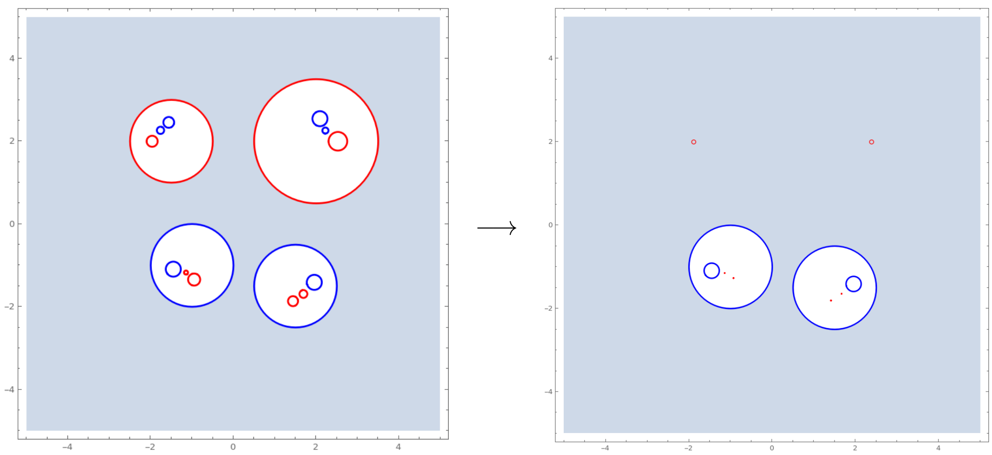

Figure 8: An analogous picture to Figure 7, but on a Schottky cover and going from genus two to genus one. The red $\mathfrak{A}$-cycle is shrunk down to a single point, revealing the Schottky cover for a genus one surface with two marked points. Note that the shrinking of the cycle leads to punctures at the fixed points P_i and P_i', which do not exactly correspond to the centers of the circles.

We must then understand what this procedure means for the corresponding Schottky group. For the concentric Schottky cover at genus one (Figure 6), this process corresponds to taking $\tau \to i\infty$, which makes $\gamma : z \mapsto 0 = P_1$ and $\gamma_1^{-1} \mapsto \infty = P_1'$. At higher genera, this procedure is typically tough to control, since it is not trivial to see how the period matrix should change when a cycle is cut. However, on the Schottky cover we have a simple prescription, as one may degenerate the surface on cutting the cycle $\mathfrak{A}_j$ by asserting that we take the limit such that

$\gamma_j : z \mapsto P_j$ and $\gamma_j^{-1} : z \mapsto P'_j$, while all other generators remain the same.

Applying this to the Kronecker forms, we find two different results depending on whether $j \stackrel{?}{=} k$ for the cycle cut $\mathfrak{A}_j$ and the Kronecker form considered F_k.

In the case that $j \neq k$, we find that the term $\gamma P_k - \gamma x$ vanishes if γ contains any appearances of γ_j, since those will map both terms to the same point. As a result, writing $\tilde{\Gamma}$ for the Schottky group generated by every generator of Γ except γ_k, we have

$$F_k(z, x, \vec{w}|\Gamma_{\gamma_j \to \infty}) = \sum_{\gamma \in \tilde{\Gamma}} \frac{dz}{z - \gamma x} \frac{\gamma P_k - \gamma x}{\gamma P_k - z} W(\gamma) = F_k(z, x, \{w_i\}_{i \neq j}|\tilde{\Gamma}), \tag{8.10}$$

recovering the corresponding Kronecker form at the reduced genus on the Schottky group corresponding to the degenerated surface.

In the case that $j = k$, we expect to have a more unconventional result, knowing that the rest of the Kronecker forms already span the whole basis. Indeed, by splitting $\gamma \in \Gamma$ into $\tilde{\gamma} \in \Gamma/\Gamma_j$ and γ_j^n $\forall n \in \mathbb{Z}$, and using properties of the fixed points, we find that

$$\begin{aligned} F_j(z, x, \vec{w}|\Gamma_{\gamma_j \to \infty}) &= \sum_{\tilde{\gamma} \in \Gamma/\Gamma_j} \sum_{n \in \mathbb{Z}} \frac{dz}{z - \tilde{\gamma}\gamma_j^n x} \frac{\tilde{\gamma}\gamma_j^n P_j - \tilde{\gamma}\gamma_j^n x}{\tilde{\gamma}\gamma_j^n P_j - z} W(\tilde{\gamma}\gamma_j^n) \\ &= \left(\sum_{\tilde{\gamma} \in \Gamma/\Gamma_j} \frac{dz}{z - \tilde{\gamma} P'_j} \frac{\tilde{\gamma} P_j - \tilde{\gamma} P'_j}{\tilde{\gamma} P_j - z} W(\tilde{\gamma}) \right) \left(\sum_{n<0} w_j^n \right) \\ &\quad + \sum_{\tilde{\gamma} \in \Gamma/\Gamma_j} \frac{dz}{z - \tilde{\gamma} x} \frac{\tilde{\gamma} P_j - \tilde{\gamma} x}{\tilde{\gamma} P_j - z} W(\tilde{\gamma}) \\ &\quad + \sum_{\tilde{\gamma} \in \Gamma/\Gamma_j} \frac{dz}{z - \tilde{\gamma} P_j} \frac{\tilde{\gamma} P_j - \tilde{\gamma} P_j}{\tilde{\gamma} P_j - z} W(\tilde{\gamma}), \end{aligned} \tag{8.11}$$

where the first term corresponds to those where $n < 0$, which turned $\gamma_j^n x \mapsto P'_j$, etc. The last term vanishes because we have the numerator $\tilde{\gamma} P_j - \tilde{\gamma} P_j = 0$. In the other two terms, we find that any element $\tilde{\gamma}$ that contains an appearance of γ_j will make the numerator similarly vanish, so we can replace the sum over Γ/Γ_j with $\tilde{\Gamma}$. Thus, we have

$$\begin{aligned} F_j(z, x, \vec{w}|\Gamma_{\gamma_j \to \infty}) &= \left(\sum_{\tilde{\gamma} \in \tilde{\Gamma}} \frac{dz}{z - \tilde{\gamma} P'_j} \frac{\tilde{\gamma} P_j - \tilde{\gamma} P'_j}{\tilde{\gamma} P_j - z} W(\tilde{\gamma}) \right) \left(\sum_{n<0} w_j^n \right) \\ &\quad + \sum_{\tilde{\gamma} \in \tilde{\Gamma}} \frac{dz}{z - \tilde{\gamma} x} \frac{\tilde{\gamma} P_j - \tilde{\gamma} x}{\tilde{\gamma} P_j - z} W(\tilde{\gamma}). \end{aligned} \tag{8.12}$$

Both of these terms look very much like Kronecker forms themselves, although they both use a fixed point which is now at the puncture left by the cut cycle. Unfortunately, interpreting implications of this degeneration limit is beyond the scope of this thesis.

8.3 Expansion into kernels

In order to expand into integration kernels, we must identify w_j with an exponential of a power-counting variable, analogously to how $w_j = \exp(-2\pi i \alpha_j)$ for the theta function representation. Following a convention similar to [Enr14, Zer23], we will simply set $w_j = \exp(b_j)$, where the letters b_j do not commute and we will identify integration kernels as coefficients of corresponding words. This choice will make our expressions simpler, and differs from the α_j notation only by a factor of $(-2\pi i)^m$ where the m is the weight of the kernel.

8.3.1 Lowest order expansion

The first step to expanding the Kronecker form is handling the lowest order term, corresponding to the term of order $\mathcal{O}(b^{-1})$. The best way to handle this is by starting with the genus one Kronecker form. From Equation 4.5 we know that[17] the weight 0 genus one kernel is $2\pi i\omega(z|\Gamma)$, which on the Schottky cover has the expression (Equation 3.6)

$$\omega(z|\Gamma_{h=1}) = \frac{1}{2\pi i}\left(\frac{1}{z-P_1} - \frac{1}{z-P_1'}\right) dz. \tag{8.13}$$

By considering the expansion given in Equation 8.9, we can find the lowest order term as an average of the $g^{(0)}$ kernel at genus one

$$\begin{aligned} F_j(z,x,\vec{w}|\Gamma) &= \sum_{\tilde{\gamma}\in\Gamma_j\backslash\Gamma} W(\tilde{\gamma}^{-1})F(\tilde{\gamma}z,x,w_j|\Gamma_j) \\ &= \sum_{\tilde{\gamma}\in\Gamma_j\backslash\Gamma}\left[\left(\frac{1}{z-P_1} - \frac{1}{z-P_1'}\right)dz + O(b^0)\right] \\ &= 2\pi i\omega_j(z|\Gamma)\frac{1}{b_j} + O(b^0), \end{aligned} \tag{8.14}$$

which is again the holomorphic differential due to the definition in Equation 3.6.

8.3.2 Full expansion

As we find the coefficients corresponding to arbitrary words $b_{i_1}\cdots b_{i_s}$, we will include a pole-cancellation factor b_j, and write

$$F(z,x,\vec{w}|\Gamma)b_j = \sum_{s=0}^{\infty}\sum_{i_1,\cdots,i_s=1}^{h} 2\pi i\omega_{i_1\cdots i_s j}(z,x|\Gamma)b_{i_1}\cdots b_{i_s} = \sum_{\gamma\in\Gamma/\Gamma_j} F_j(\gamma^{-1}z,x,w_j|\Gamma_j)W(\gamma), \tag{8.15}$$

which matches the notation above at lowest order.

In searching for an expression for these $\omega_{\cdots j}$, we will once again turn to the genus one case to give us control over the pole. This leaves us with a combinatorics problem where we need to analyze which terms in the series defining the Kronecker form have contributions to the word $b_{i_1}\cdots b_{i_s}$, taking into account the coefficients that come out of the exponentials. It is best to group our word with repeated letters, writing $b_{i_1}^{n_1}\cdots b_{i_s}^{n_s}$ where $i_k \neq i_{k+1}$. With this in mind, we can keep track of contributions from elements where γ_{i_1} appears $m_1 \leq n_1$ times, followed by γ_{i_2} appearing $m_2 \leq n_2$ times, etc. with any generators appearing in between them that do not contribute. To avoid double counting for different values of m_1, we assert that each of the generators we have singled out must contribute at least a single letter, so the unit term from the exponentials should be ignored. As such, the corresponding exponentials are

$$\cdots\underbrace{\gamma_{i_1}\cdots\gamma_{i_1}}_{\geq m_1\text{appearances of}\gamma_{i_1}}\cdots\overbrace{\gamma_{i_s}\cdots\gamma_{i_s}}^{\geq m_s\text{appearances of}\gamma_{i_s}}\cdots \to (e^{b_{i_1}}-1)^{m_1}\cdots(e^{b_{i_s}}-1)^{m_s}, \tag{8.16}$$

[17] One must be careful with the powercounting variables. In Section 4.2.2, we were using $w = e^{-2\pi i\alpha}$, and $\alpha F = \sum_{n=0}^{\infty}\alpha^n g_\alpha^{(n)}$. However, in the current section, we have $w = e^b$, and $Fb = \sum_{n=0}^{\infty} b^n g_b^{(n)}$. Identifying $b = -2\pi i\alpha$, we find $g_b^{(0)} = 2\pi i g_\alpha^{(0)}$. From Equation 4.5, we had $g_\alpha^{(0)} = \omega$, so in the current language with expansions in b, we must have the weight 0 kernel be $2\pi i\omega$.

from which we must extract the coefficient corresponding to the word $b_{i_1}^{n_1}\cdots b_{i_s}^{n_s}$. We will write the collection of elements $\gamma\in\Gamma/\Gamma_j$ of that form as $\Gamma(\vec{m})$, keeping in mind that the same element may appear multiple times due to the generators appearing in the omitted '$\cdots$' sections[18].

We can solve the simple case of extracting b^n from $(e^b-1)^m$, finding

$$\begin{aligned}(e^b-1)^m&=\sum_{k=0}^{m}e^{kb}(-1)^{m-k}\frac{m!}{k!(m-k)!}=\sum_{k=0}^{m}\sum_{j=0}^{\infty}\frac{(kb)^j}{j!}(-1)^{m-k}\\&=\cdots+\left(\sum_{k=0}^{m}\frac{k^n m!(-1)^{m-k}}{n!k!(m-k)!}\right)b^n+\cdots,\end{aligned}\tag{8.17}$$

where we label $C(m,n)=\sum_{k=0}^{m}\frac{k^n m!(-1)^{m-k}}{n!k!(m-k)!}$.

Then, the coefficient corresponding to $b_{i_1}^{n_1}\cdots b_{i_s}^{n_s}$ from $(e^{b_{i_1}}-1)^{m_1}\cdots(e^{b_{i_s}}-1)^{m_s}$ is $C(\vec{m},\vec{n})=\prod_{j=1}^{s}C(m_j,n_j)$. As we apply this result to an explicit formula for $\omega_{\dots j}$, we also recognize that the genus one Kronecker form $F_j(z,x,w_j|\Gamma_j)$ can also contribute letters b_j by using higher weight kernels, which matters if $i_s=j$. Putting it all together, we have

$$\omega_{\underbrace{i_1\cdots i_1}_{n_1\text{ times}}\cdots\underbrace{i_s\cdots i_s}_{n_s\text{ times}}j}=\frac{1}{2\pi i}\sum_{M=0}^{\delta_{i_s j}n_s}\sum_{m_1=1}^{n_1}\cdots\sum_{m_s=1}^{n_s-M}\sum_{\gamma\in\Gamma(\vec{m})}C(\vec{m},\vec{n}-M\vec{\delta}_s)g_b^{(M)}(\gamma^{-1}z,x|\Gamma_j),\tag{8.18}$$

giving us an explicit, albeit tedious, expression for higher genus kernels as averages of genus one kernels on a subcover[19].

8.3.3 Relationship to Enriquez' connection K

With our basis of Kronecker forms, we may combine them into one larger object that contains all the generated differentials

$$\tilde{K}(z,x)=\sum_{j=1}^{h}\frac{1}{2\pi i}F_j(z,x,\vec{w}|\Gamma)b_ja_j=\sum_{s=0}^{\infty}\sum_{i_1,\cdots,i_s=1}^{h}\sum_{j=1}^{h}\omega_{i_1\cdots i_s j}(z,x)b_{i_1}\cdots b_{i_s}a_j,\tag{8.19}$$

where s, the length of the word in b's, corresponds to the weight of the corresponding coefficient, which is consistent with $\omega_1=g^{(0)}$ at genus one. Note that this object includes factors b_j that cancel the poles in the variable b_j that each F_j contains.

Due to the quasiperiodicity and residue of the Kronecker forms (Section 8.2.1), we find that this one-form satisfies

$$\tilde{K}(\gamma_k z,x)=e^{b_k}\tilde{K}(z,x)\quad;\quad \mathrm{res}_{z=x}\tilde{K}(z,x)=\sum_{j=1}^{h}b_ja_j dz/2\pi i,\tag{8.20}$$

precisely the conditions specified by [EZ21] that uniquely determine the Enriquez' connection K, provided that one restricts to one-forms without negative powers of b_j[20].

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[18]e.g. $\Gamma(1,1)$ includes $\gamma_1\gamma_1\gamma_2$ twice since it appears as $\cdots\gamma_1\gamma_2$ and as $\gamma_1\cdots\gamma_2$.

[19]Where as in Footnote 17, one must be careful to use $bF(z,x,w|\Gamma_{h=1})=\sum_{n=0}^{\infty}b^n g_b^{(n)}(z,x)$, rather than the form with α. Since b is identified with $-2\pi i\alpha$, using $(-2\pi i)^{n-1}g_b^{(n)}=-g_\alpha^{(n)}$ one may reuse existing formulas, such as the q-expansions presented in Section 4.2.2.

[20]Without this restriction, the function $K=F_1\sum_{j=1}^{h}b_ja_j$ would in principle satisfy the same conditions. However, $F_1=\omega_1/b_1+\cdots$ so K would include some words with $1/b_1$ in them.

We can make a direct connection to the original paper [Enr14] by studying the properties of the coefficients $\omega_{\dots}$ directly, finding

$$\begin{aligned}\tilde{K}(\gamma_k z, x) = e^{b_k}\tilde{K}(z,x) &\implies \omega_{i_1\cdots i_s j}(\gamma_k z, x) = \sum_{l=0}^{s} \delta_{i_1\cdots i_l k}\omega_{i_{l+1}\cdots i_s j}(z,x) \\ \text{res}_{z=x}\tilde{K}(z,x) = \sum_{j=1}^{h} b_j a_j dz &\implies \text{res}_{z=x}\omega_{i_1\cdots i_s j}(\gamma_k z, x) = \delta_{s1}\delta_{i_1 j}dz.\end{aligned} \tag{8.21}$$

As at genus one, the kernels have an additive quasiperiodicity where kernels of lower weight are added to them, and the only kernels that contain poles are those that are of weight 1. These results, as well as recognizing $\omega_j(z,x) = \omega_j(z)$, match the constraints on ω given by Enriquez in [Enr14, Lemma 6].

9 Conclusion

9.1 Open questions

9.1.1 Relationship to twisted Green function

The Kronecker form defined as a ratio of theta functions is remarkably similar to the twisted Green function defined by Enriquez and Felder in [EF00, Equation 11]. To demonstrate the similarity it is useful to define a theta function with a particular shift $\theta_{\text{EF}} : \vec{v} \mapsto \Theta(\vec{v} - (h-1)\mathfrak{u}(P_0) - \vec{K})$, where $\vec{K}$ is the vector of Riemann constants[21]. In particular, since $\Theta(-\mathfrak{u}(D) - \vec{K}) = 0$ for divisors of degree $h-1$, we find that $\theta_{\text{EF}}(\vec{0})$ vanishes at the origin, just like the theta functions used to construct the Kronecker form defined in Equation 7.2. With these conventions, the twisted Green function of Enriquez and Felder is

$$G(z,x,\vec{\alpha}) = \frac{\theta_{\text{EF}}(\mathfrak{u}(x-z)+\vec{\alpha})}{\theta_{\text{EF}}(\mathfrak{u}(x-z))\theta_{\text{EF}}(\vec{\alpha})} d_z(\theta_{\text{EF}}(\mathfrak{u}(z)-\mathfrak{u}(p)))\Big|_{p=z}, \tag{9.1}$$

almost exactly like the Kronecker form up to a few signs. The relationship between these objects may be explored further to understand how the Kronecker form as a ratio of theta functions is related to the KZB connection constructed by Enriquez and Felder.

9.1.2 Periodic analogue of the Schottky representation

It may be desirable for some applications to find integration kernels that are completely periodic. This was relatively easy to accomplish in the commutative case with the theta function representation, since we can readily use exponentials to find quasiperiodic functions

$$f(z) = \exp\left(2\pi i \sum_{i,j} Y_{ij}\text{Im}(\mathfrak{u}_i(z))\alpha_j\right) \implies f(z+\mathfrak{B}_j) = \exp(2\pi i\alpha_j)f(z), \tag{9.2}$$

as was done in Section 7.2.5.

[21]The conventions used in [EF00] differ from those in this thesis, which we defined in Equation 2.19. Instead, they refer to $\Delta = -\vec{K}$. The definition of $G(z,x,\vec{\alpha})$ below is modified to use the conventions from this thesis. Matching the definition from [EF00] requires using $\Theta(\vec{v}) = \Theta(-\vec{v})$ to invert some of the signs, and identifying w with x and $\vec{\lambda}$ with $-\vec{\alpha}$.

However, the same strategy cannot be used in the non-commutative case. One must instead find some function $f(z,\vec{w}|\Gamma)$ satisfying $f(\gamma_k z,\vec{w}|\Gamma) = f(z,\vec{w}|\Gamma)w_k^{-1}$, so that the form,

$$\Omega(z,x,\vec{w}|\Gamma) = f(z,\vec{w}|\Gamma)F(z,x,\vec{w}|\Gamma), \tag{9.3}$$

satisfies

$$\Omega(\gamma_k z,x,\vec{w}|\Gamma) = f(\gamma_k z,\vec{w}|\Gamma)F(\gamma_k z,x,\vec{w}|\Gamma) = f(z,\vec{w}|\Gamma)w_k^{-1}w_kF(z,x,\vec{w}|\Gamma) = \Omega(z,x,\vec{w}|\Gamma). \tag{9.4}$$

In principle, it is possible to write $f(z,\vec{w}|\Gamma) = \sum_{\gamma\in\Gamma}\tilde{f}(\gamma z)W(\gamma)$. However, naive guesses for a Moebius invariant $\tilde{f}$ like

$$\tilde{f}(z|\Gamma) = \frac{(z-P_i)(P_j-P_k)}{(z-P_j)(P_i-P_k)}, \tag{9.5}$$

will lead to a divergent series since the terms $(\gamma P_j - \gamma P_k)$ and $(\gamma P_i - \gamma P_k)$ shrink at approximately equal rates for elements with a lot of generators.

It may be possible to find an alternative ansatz for a periodicity-restoring factor, or to use a different starting point using manifestly periodic objects as in [DHS23].

9.1.3 Connection between theta and Schottky representations of Kronecker forms

In order to make a connection between the representations in Section 7 and Section 8, we would like to be able to either make the representation in terms of theta functions non-commutative, or make the representation on the Schottky cover commutative.

If we start with the Schottky representation (Equation 8.2), we must find component functions $f_{\vec{n}}(z)$ corresponding to $f(z|\Gamma)$ such that

$$\sum_{\gamma\in\Gamma} W(\gamma^{-1})f(\gamma z) = \sum_{\vec{n}\in\mathbb{Z}}\left(\prod_{j=1}^{h} w_j^{-n_j}\right)f_{\vec{n}}(z), \tag{9.6}$$

where $f_{\vec{n}}(z+\mathfrak{B}_j,\cdots) = hf_{\vec{n}+\vec{\delta}_j}(z)$, for $(\vec{\delta}_j)_i = \delta_{ij}$, and $W : \gamma_{i_1}^{n_1}\cdots\gamma_{i_s}^{n_s} \mapsto w_{i_1}^{n_1}\cdots w_{i_s}^{n_s}$. By matching powers of w_j, we find that

$$f_{\vec{n}}(z) = \sum_{\substack{\gamma\in\Gamma \\ \mathrm{ord}_j(\gamma)=n_j\ \forall i}} f(\gamma z), \tag{9.7}$$

where $\mathrm{ord}_j : \gamma_{i_1}^{n_1}\cdots\gamma_{i_s}^{n_s} \mapsto \sum_{k=1}^{s}\delta_{ji_k}n_k$. This restriction to elements containing precisely the right generators still leaves us with an infinite sum, and makes it difficult to compare the representations. Going the other way, from commutative to non-commutative, requires not only finding $f_{\vec{n}}$ corresponding to the theta Kronecker forms (Equation 7.2), but guessing a sum over these infinite subsets of Γ.

Without finding explicit ways to switch the commutativity of the representations, we may nonetheless hope to find a connection by expressing one representation as a linear combination of the other. For example, due to their convenience, we may use the Schottky Kronecker forms as the basis. Then, imposing $w_j = \exp(-2\pi\alpha_j)$, one would expect that we may write $F(z,x,\vec{\alpha}) = \sum_{j=1}^{h} c_j(x,\vec{\alpha})F_j(z,x,\vec{w})$.

9.1.4 Generalizing Fay identities

Using the representation as a ratio of theta functions, it is clear that at arbitrary genus there are at least $2^{h-1}(2^h - 1)$ Kronecker forms[22] that satisfy Fay identities of the form

$$F(b, a, \vec{\alpha})F(d, a, \vec{\beta}) = F(b, d, \vec{\alpha})F(d, a, \vec{\alpha} + \vec{\beta}) + F(b, a, \vec{\alpha} + \vec{\beta})F(d, b, \vec{\beta}). \tag{9.8}$$

Though their use for integration kernels is restricted by the requirement that $(\vec{\alpha}, \vec{\beta}) = (\vec{x}, \mathfrak{u}(c) - \mathfrak{u}(d))$ for some $x \in \mathbb{C}^h$ and $c \in \mathcal{M}$, it is nonetheless important to consider how these Fay identities may be identified for a larger family of quasiperiodic differential forms.

Working with the Schottky representation, one can actually easily reproduce the Fay identity at genus one. By working with a concentric Schottky cover where $\gamma_1 : z \mapsto qz$ and $P_1 = \infty$, we have $F(z, x, w) = \sum_{n\in\mathbb{Z}} \frac{dz}{z-q^n x} w^{-n}$ (Equation 4.17 and [Cha22]) and we find

$$\begin{aligned}
F(z, x, w)F(\tilde{z}, \tilde{x}, \tilde{w}) &= \sum_{n\in\mathbb{Z}} \frac{dz}{z - q^n x} w^{-n} \sum_{\tilde{n}\in\mathbb{Z}} \frac{d\tilde{z}}{\tilde{z} - q^{\tilde{n}}\tilde{x}} \tilde{w}^{-\tilde{n}} \\
&= \sum_{n\in\mathbb{Z}} \sum_{\tilde{n}\in\mathbb{Z}} \left(\frac{dz}{z - q^n x} \frac{d\tilde{z}/z}{\tilde{z}/z - q^{\tilde{n}-n}\tilde{x}/x} + [\tilde{\cdot} \leftrightarrow \cdot] \right) w^{-n} w^{-\tilde{n}} \\
&= F(z, x, w\tilde{w})F(\tilde{z}/z, \tilde{x}/x, w) + F(\tilde{z}, \tilde{x}, w\tilde{w})F(z/\tilde{z}, x/\tilde{x}, \tilde{w}),
\end{aligned} \tag{9.9}$$

precisely like Equation 4.11 by noticing how on the concentric Schottky cover we have $\mathfrak{u}(a/b) = \mathfrak{u}(a) - \mathfrak{u}(b)$. However, attempting to generalize this to higher genus on the Schottky cover causes into several issues. The key points of difficulty in the derivation are the fact that fixed points are no longer invariant under most elements of the Schottky group, and that there isn't a clear geometric identification for the difference of the images of Abel maps. Furthermore, the process of finding a Fay identity is fundamentally challenging with non-commutative monodromy factors, since one must not only match the quantity of monodromy factors but also their positions within the words with non-commuting letters.

An alternative approach to generalizing Fay identities relies on better understanding the way that coefficients must be structured when using the theta Kronecker forms as a basis. In principle, by writing $\tilde{F}(z, x, \vec{\alpha}) = \sum_{i=1}^h c_i(x, \vec{\alpha})F_i(z, x, \vec{\alpha})$ for odd Kronecker forms F_i, one may be able to expand $\tilde{F}\tilde{F}$ and use the Fay identities for F_i to find a corresponding Fay identity for the full $\tilde{F}$.

9.1.5 Representation using modular forms

This thesis did not tackle the question of finding a representation of the Kronecker form using something analogous to the Eisenstein series and functions (Section 4.2.3). One may consider working with higher genus analogues of these objects, such as the Siegel modular forms from [BvdGHZ08]. This representation has an advantage compared to the theta ratio and Schottky cover since the modular forms manifestly give a prescription for how the Kronecker function, and the corresponding integration kernels, transform when we transform the period matrix.

By adding $\mathfrak{A}$-cycles to $\mathfrak{B}$-cycles, or switching the labeling of the two, the modular parameter at genus one are invariant under $\tau \mapsto \tau + 1$ and $\tau \mapsto -1/\tau$, the generators of the $\mathrm{SL}(2, \mathbb{Z})$. The Eisenstein series

$$e_j(\tau) = \sum_{\substack{m,n\in\mathbb{Z} \\ (m,n)\neq(0,0)}}^{L} \frac{1}{(m + n\tau)^j}, \tag{9.10}$$

[22] Corresponding to the number of odd theta functions, as mentioned in Section 2.3.1.

is a modular form with respect to such transformation. This means that for $\tau' = \frac{a\tau+b}{c\tau+d}$, we have $e_j(\tau') = (c\tau+d)^j e_j(\tau)$.

Similar transformation properties are true for the period matrix at higher genera using $\mathrm{Sp}(2h,\mathbb{Z})$ at genus h, and with the corresponding modular forms being the aforementioned Siegel modular forms. Though the precise definition of these modular forms is beyond the scope of this thesis, it is nonethleess possible to point out a few challenges with them. Unlike the Eisenstein series above, defined over the entire lattice except the origin, Siegel modular forms are defined over a particular subgroup of relatively prime matrices. This would be analogous to the notation at genus one

$$G_j(\tau) = \sum_{\substack{m,n\in\mathbb{Z} \\ (m,n)\neq(0,0) \\ \nexists a\neq 1 \text{ s.t. } a|m \text{ and } a|n}} \frac{1}{(m+n\tau)^j}, \tag{9.11}$$

which is related to the Eisenstein series above by identifying

$$e_j(\tau) = \sum_{\substack{m,n\in\mathbb{Z} \\ (m,n)\neq(0,0)}}^{L} \frac{1}{(m+n\tau)^j} = \sum_{k=1}^{\infty} \sum_{\substack{m,n\in\mathbb{Z} \\ (m,n)\neq(0,0) \\ \nexists a\neq 1 \text{ s.t. } a|m \text{ and } a|n}} \frac{1}{(km+kn\tau)^j} = \sum_{k=1}^{\infty} \frac{1}{k^j} G_j(\tau) = \zeta(j) G_j(\tau). \tag{9.12}$$

With the difficulties faced with finding an analogue of this conversion for matrices, it becomes difficult to identify an analogue of the Eisenstein function

$$E_j(z,\tau) = \lim_{L\to\infty} \sum_{m,n=-L}^{L} \frac{1}{(z+m+n\tau)^j}, \tag{9.13}$$

which takes advantage of the sum over the full lattice to average over shifts of a point z.

One strategy to approach generalizations to higher genus may be studying the relationship between the Eisenstein function and logarithmic derivatives of the odd theta function [BL11], or the prime form[23], $E_1(z,\tau) = \partial_z \ln(\theta(z)) = \partial_z \ln(E(z,0))$.

9.1.6 Connections to algebraic curves

An approach to identifying integration kernels for polylogarithms not yet mentioned in this thesis is the one described in Section 3 of [BDDT18], where integration kernels are defined directly on the corresponding elliptic curve. Algebraic curves relevant to the study of higher genus Riemann surfaces can be defined through equations of the form

$$y^2 = P_{2h+1}(x) = (x-a_1)\cdots(x-a_{2h+1}), \tag{9.14}$$

where the square on the left hand side produces a set of solutions that lives on two sheets, which are connected by branch cuts connecting the points a_i. The basis of holomorphic differentials on algebraic curves is [Bob11]

$$\omega_j(x,y) = \frac{x^{j-1}dx}{y}, \tag{9.15}$$

[23]It may be unclear how the derivative of the logarithm of the prime form should act given the half-differentials it includes. However, the differentiation kills these terms at genus one, leaving only the odd theta function from the numerator.

and at genus one, one can identify higher weight kernels, such as the weight 1 kernels[24]

$$\varphi_1(x,c) = \frac{1}{x-c}\frac{dx}{y} \quad ; \quad \varphi_{-1}(x,c) = \frac{\sqrt{P_3(c)}}{y(x-c)}\frac{dx}{y}, \tag{9.16}$$

each of which have simple poles at $(x, y) = (c, \pm\sqrt{P_3(c)})$. These kernels on the algebraic curve, and their higher weight analogues, are related to linear combinations of the integration kernels $g^{(n)}$ (defined in Section 4.2).

One may anticipate a similar relationship to exist with the kernels $g^{(\vec{n})}$ from the theta Kronecker forms, or to the kernels $\omega_{\cdots j}$ from the Schottky Kronecker form.

9.2 Summary and outlook

Throughout this thesis, we balanced several approaches to the consideration of functions and differential forms on Riemann surfaces of genus h.

This started with Section 2.1, where we dealt with compact Riemann surfaces as abstract spaces with local charts and a structure coming from their homotopy group and holomorphic differentials. This language, without necessarily referring to specific representations, is sufficient to define what we mean by Kronecker forms, identifying them by their quasiperiodicity and residue, as was done in Section 5. In Section 6, we studied the space of quasiperiodic forms, proving that the space of quasiperiodic forms with a simple pole is of dimenstion h, implying that there are only h independent Kronecker forms at genus h. In anticipation of this conclusion, we have uniquely identified the genus one Kronecker function by its quasiperiodicity and residue in Section 4.2. With this conclusion, we could begin work using specific coordinates on the Riemann surface to find explicit representations of the bases of Kronecker forms.

The first approach to explicit representations was built upon the Abel map in Section 2.2. By mapping the manifold to $\mathbb{C}^h$, with the abelianized homotopy group corresponding to a lattice on the space, we defined theta functions that work naturally with this lattice and studied their zeros. After seeing how a ratio of odd theta functions defines the Kronecker function at genus one in Section 4.2.1, we generalized the procedure to arbitrary genus Kronecker forms in Section 7. These Kronecker forms span the corresponding space by selecting different theta functions for the ratios and using linear combinations of the results. Aside from satisfying a higher genus Fay identity, and being related to the generalization suggested by [Tsu23], these Kronecker forms can be expanded by applying some pole-cancellation functions.

The other approach concerned itself with finding an expression on a Schottky cover, defined in Section 3. By identifying a suitable finitely-generated subgroup of Moebius transformations, a higher genus Riemann surface can be identified with $\mathfrak{B}$-cycles corresponding to actions of the generators, and $\mathfrak{A}$-cycles to the circles they map to each other. Section 4.2.2 explored how the second definition at genus one given by [BL11] can be rewritten on a concentric Schottky cover [Cha22], and imbued with Moebius invariance to work on an arbitrary Schottky cover. In Section 8, the resulting formula is straightforwardly generalized to higher genera. By selecting the fixed points corresponding to the generators of the Schottky group for the factor restoring Moebius invariance, the Kronecker forms on the Schottky cover form a neat, well-normalized basis for quasiperiodic forms. Working through the combinatorics for the non-commuting power-counting variables, we can find an explicit expression for the corresponding integration kernels. By shrinking an $\mathfrak{A}$-cycle to a point, the Schottky group degenerates to lower genus, and the Kronecker forms degenerate directly to a basis of the corresponding surface, with an extra element

[24] Note that the notation here differs from [BDDT18], since we continue to follow the convention of identifying kernels directly with the corresponding one-forms. At genus one, this corresponds to multiplying by ω_1.

containing poles at the punctures. Finally, this basis of Schottky Kronecker forms has a direct relationship with Enriquez' connection [Enr14, EZ21], as it describe an explicit representation of the components.

Using these two results, we conclude that the conditions of quasiperiodicity on $\mathfrak{B}$-cycles and unit residue are a suitable starting point for defining kernels for defining polylogarithms. The representations in terms of theta functions and on the Schottky cover have a key difference in the commutativity of the power-counting variables, and each has several properties that don't seem to be easily spotted in the other representation, such as the Fay identity from the theta ratio and the degeneration limits from the Schottky sum. It is possible that studying similar constructions of Kronecker forms using alternative languages, such as on the universal cover with the Fuchsian group, on an algebraic curve, or using modular forms, we will be able to reproduce these properties, find new ones, or make room for connections between the commutative and non-commutative approaches. As higher genus polylogarithms are developed and understood, one can freely choose a language which contains the most convenient properties, knowing that they are equivalent to other representations by the unique characterization.

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Appendices

A Combinatorics for the genus one Fay identity

Starting with the Fay identity for the Kronecker function (Equation 4.11), we multiply by $(\alpha)(\tilde{\alpha})(\alpha+\tilde{\alpha})$

$$(\alpha+\tilde{\alpha})[\alpha F(z,\alpha)][\tilde{\alpha}F(\tilde{z},\tilde{\alpha})] = \tag{A.1}$$

$$\alpha[(\alpha+\tilde{\alpha})F(z,\alpha+\tilde{\alpha})][\tilde{\alpha}F(\tilde{z}-z,\tilde{\alpha})] + \tilde{\alpha}[(\alpha+\tilde{\alpha})F(\tilde{z},\alpha+\tilde{\alpha})][\alpha F(z-\tilde{z},\alpha)], \tag{A.2}$$

which allows us to expand each term in the kernels g

$$(\alpha+\tilde{\alpha})\sum_{j=0}^{\infty}\alpha^j g^{(j)}(z)\sum_{k=0}^{\infty}\tilde{\alpha}^k g^{(k)}(\tilde{z}) = \tag{A.3}$$

$$\alpha\sum_{j=0}^{\infty}(\alpha+\tilde{\alpha})^j g^{(j)}(z)\sum_{k=0}^{\infty}\tilde{\alpha}^k g^{(k)}(\tilde{z}-z)+ \tag{A.4}$$

$$\tilde{\alpha}\sum_{j=0}^{\infty}(\alpha+\tilde{\alpha})^j g^{(j)}(\tilde{z})\sum_{k=0}^{\infty}\alpha^k g^{(k)}(z-\tilde{z}) \tag{A.5}$$

Isolating the term corresponding to $\alpha^{m+1}\tilde{\alpha}^n$, we find

$$g^{(m)}(z_1)g^{(n)}(z_2)+g^{(m+1)}(z_1)g^{(n-1)}(z_2) = \tag{A.6}$$

$$\sum_{j=0}^{n}\binom{m+j}{j}g^{(m+j)}(z_1)g^{(n-j)}(z_2-z_1)+ \tag{A.7}$$

$$\sum_{j=0}^{m+1}\binom{n-1+j}{j}g^{(n-1+j)}(z_2)g^{(m+1-j)}(z_1-z_2), \tag{A.8}$$

To demonstrate the next step, we will write the left and right sides of the above equation as $\sigma_L(m,n)$ and $\sigma_R(m,n)$ respectively,

$$\sigma_L(m,n)=\sigma_R(m,n). \tag{A.9}$$

Then, we can study the equality

$$\sum_{k=0}^{n-1}(-1)^k\sigma_L(m+k,n-k)=\sum_{k=0}^{n-1}(-1)^k\sigma_R(m+k,n-k). \tag{A.10}$$

For the left side of Equation A.10, we find that this results in a telescoping series, in which most terms end up cancelling

$$\sum_{k=0}^{n-1}(-1)^k\sigma_L(m+k,n-k) = \tag{A.11}$$

$$\left(g^{(m)}(z_1)g^{(n)}(z_2)+\cancel{g^{(m+1)}(z_1)g^{(n-1)}(z_2)}\right) \tag{A.12}$$

$$-\left(\cancel{g^{(m+1)}(z_1)g^{(n-1)}(z_2)}+\cancel{g^{(m+2)}(z_1)g^{(n-2)}(z_2)}\right) \tag{A.13}$$

$$+\cdots+(-1)^{m-1}\left(\cancel{g^{(m+n-1)}(z_1)g^{(1)}(z_2)}+g^{(m+n)}(z_1)g^{(0)}(z_2)\right) \tag{A.14}$$

$$=g^{(m)}(z_1)g^{(n)}(z_2)+(-1)^{n-1}g^{(m+n)}(z_1), \tag{A.15}$$

and the last term is simplified using $g^{(0)}(z_2)=1$.

Thus, we are able to isolate the $g^{(m)}(z_1)g^{(n)}(z_2)$ term as

$$g^{(m)}(z_1)g^{(n)}(z_2)=(-1)^n g^{(m+n)}(z_1)+\sum_{k=0}^{n-1}(-1)^k\sigma_R(m+k,n-k). \tag{A.16}$$

Studying the right side,

$$(-1)^n g^{(m+n)}(z_1) + \sum_{k=0}^{n-1}(-1)^k \sigma_R(m+k, n-k) = \tag{A.17}$$

$$\sum_{k=0}^{n}(-1)^k \sum_{j=0}^{n-k} \binom{m+j+k}{j} g^{(m+j+k)}(z_1) g^{(n-j-k)}(z_2 - z_1) + \tag{A.18}$$

$$\sum_{k=0}^{n-1}(-1)^k \sum_{j=0}^{m+k+1} \binom{n+j-k-1}{j} g^{(n+j-k-1)}(z_2) g^{(m-j+k+1)}(z_1 - z_2), \tag{A.19}$$

where we absorbed the extra $g^{(m+n)}(z_2)$ term into the sum over k in the second line.

By defining $r = j + k$, we rearrange the dependencies in line A.18 such that the functions g can be taken out of one of the sums

$$\sum_{k=0}^{n}(-1)^k \sum_{j=0}^{n-k} \binom{m+j+k}{j} g^{(m+j+k)}(z_1) g^{(n-j-k)}(z_2 - z_1) = \tag{A.20}$$

$$\sum_{r=0}^{n} g^{(m+r)}(z_1) g^{(n-r)}(z_2 - z_1) \sum_{j=0}^{r} (-1)^{r-j} \binom{m+r}{j}. \tag{A.21}$$

The remaining factor can be evaluated explicitly, by noting that $\binom{A}{B} = \binom{A-1}{B-1} + \binom{A-1}{B}$,

$$\sum_{j=0}^{r}(-1)^{r-j}\binom{m+r}{j} = \sum_{j=0}^{r}(-1)^{r-j}\left[\binom{m+r-1}{j-1} + \binom{m+r-1}{j}\right] = \binom{m+r-1}{r}, \tag{A.22}$$

where the first term was $\binom{m+r-1}{-1} = 0$, intermediate terms were cancelled by the alternating sum, and the last term survived.

Similarly, for line A.19, we can define $r = j - k - 1$ such that

$$\sum_{k=0}^{n-1}(-1)^k \sum_{j=0}^{m+k+1} \binom{n+j-k-1}{j} g^{(n+j-k-1)}(z_2) g^{(m-j+k+1)}(z_1 - z_2) = \tag{A.23}$$

$$\underbrace{(-1)^{n-1} g^{(0)}(z_2) g^{(m+n)}(z_1 - z_2)}_{r=-n,k=n-1,j=0} + \sum_{r=-n+1}^{m} g^{(n+r)}(z_2) g^{(m-r)}(z_1 - z_2) \sum_{j=r}^{n+r} (-1)^{j-r+1} \binom{n+r}{j}. \tag{A.24}$$

Again, we can focus on just simplifying the combinatoric sum, starting with $j \mapsto n + r - j$

$$\sum_{j=r}^{n+r-1} (-1)^{j-r+1} \binom{n+r}{j} = \sum_{j=0}^{n-1} (-1)^{n-j+1} \binom{n+r}{j} = \binom{n+r-1}{n-1} = \binom{n+r-1}{r}, \tag{A.25}$$

where choose functions with negative arguments are 0, restricting our sum over r to only take non-negative values.

Thus, combining all the results we find

$$g^{(m)}(z_1)g^{(n)}(z_2) = (-1)^{n-1}g^{(m+n)}(z_1 - z_2) \tag{A.26}$$

$$\sum_{r=0}^{n} \binom{m+r-1}{r} g^{(m+r)}(z_1)g^{(n-r)}(z_2 - z_1) \tag{A.27}$$

$$\sum_{r=0}^{m} \binom{n+r-1}{r} g^{(n+r)}(z_2)g^{(m-r)}(z_1 - z_2), \tag{A.28}$$

as desired.

B Proof of Fay identity at higher genus

A known property of the Theta functions is the Fay trisecant identity [Mum84]. For points a, b, c, d on a genus h compact Riemann surface, and a vector $\vec{z} \in \mathbb{C}^h$ we find

$$\begin{aligned}
&\Theta(\vec{z} + \mathfrak{u}(c) - \mathfrak{u}(a))\Theta(\vec{z} + \mathfrak{u}(d) - \mathfrak{u}(b))\theta(\mathfrak{u}(c) - \mathfrak{u}(b))\theta(\mathfrak{u}(a) - \mathfrak{u}(d)) \\
+&\Theta(\vec{z} + \mathfrak{u}(c) - \mathfrak{u}(b))\Theta(\vec{z} + \mathfrak{u}(d) - \mathfrak{u}(a))\theta(\mathfrak{u}(c) - \mathfrak{u}(a))\theta(\mathfrak{u}(d) - \mathfrak{u}(b)) \\
=&\Theta(\vec{z} + \mathfrak{u}(c) + \mathfrak{u}(d) - \mathfrak{u}(a) - \mathfrak{u}(b))\Theta(\vec{z})\theta(\mathfrak{u}(c) - \mathfrak{u}(d))\theta(\mathfrak{u}(a) - \mathfrak{u}(b)),
\end{aligned} \tag{B.1}$$

where Θ is the theta function with no characteristics and θ is an arbitrary odd non-singular theta function.

Recall that $\theta(\vec{x}) = \exp(2\pi i[\epsilon_j\tau_{jk}\epsilon_k/2 + \epsilon_j z_j + \epsilon_j\epsilon'_j])\Theta(\vec{z} + \epsilon'/2 + \tau\epsilon/2)$ (Equation 2.10) where ϵ, ϵ' are the characteristics of θ, lets us connect theta functions with and without characteristics. Consequently, relabeling $\vec{z} = \vec{y} + \vec{\eta} + \tau\vec{\epsilon}$, and cancelling out the identitical exponentials on every term, we find

$$\begin{aligned}
&\theta(\vec{y} + \mathfrak{u}(c) - \mathfrak{u}(a))\theta(\vec{y} + \mathfrak{u}(d) - \mathfrak{u}(b))\theta(\mathfrak{u}(c) - \mathfrak{u}(b))\theta(\mathfrak{u}(a) - \mathfrak{u}(d)) \\
+&\theta(\vec{y} + \mathfrak{u}(c) - \mathfrak{u}(b))\theta(\vec{y} + \mathfrak{u}(d) - \mathfrak{u}(a))\theta(\mathfrak{u}(c) - \mathfrak{u}(a))\theta(\mathfrak{u}(d) - \mathfrak{u}(b)) \\
=&\theta(\vec{y} + \mathfrak{u}(c) + \mathfrak{u}(d) - \mathfrak{u}(a) - \mathfrak{u}(b))\theta(\vec{y})\theta(\mathfrak{u}(c) - \mathfrak{u}(d))\theta(\mathfrak{u}(a) - \mathfrak{u}(b)),
\end{aligned} \tag{B.2}$$

where a factor of $\exp(2k + f(2\vec{y} + \mathfrak{u}(c + d - a - b)))$ was divided out.

Now, choosing $\vec{y} = \vec{x} - \mathfrak{u}(c) - \mathfrak{u}(d)$ and using the odd theta functions, we find

$$\begin{aligned}
&\theta(\vec{x} + \mathfrak{u}(d) + \mathfrak{u}(a))\underline{\theta(\vec{x} + \mathfrak{u}(c) + \mathfrak{u}(b))}\theta(\mathfrak{u}(c) - \mathfrak{u}(b))\underline{\theta(\mathfrak{u}(a) - \mathfrak{u}(d))} \\
+&\underline{\theta(\vec{x} + \mathfrak{u}(d) + \mathfrak{u}(b))}\theta(\vec{x} + \mathfrak{u}(c) + \mathfrak{u}(a))\theta(\mathfrak{u}(c) - \mathfrak{u}(a))\underline{\theta(\mathfrak{u}(d) - \mathfrak{u}(b))} \\
=&\theta(\vec{x} + \mathfrak{u}(a) + \mathfrak{u}(b))\theta(\vec{x} + \mathfrak{u}(c) + \mathfrak{u}(d))\underline{\theta(\mathfrak{u}(c) - \mathfrak{u}(d))\theta(\mathfrak{u}(a) - \mathfrak{u}(b))}.
\end{aligned} \tag{B.3}$$

We can divide through by the theta functions underlined above,

$$\begin{aligned}
&\frac{\theta(\vec{x} + \mathfrak{u}(d) + \mathfrak{u}(a))}{\theta(\vec{x} + \mathfrak{u}(d) + \mathfrak{u}(b))\theta(\mathfrak{u}(a) - \mathfrak{u}(b))}\frac{\theta(\mathfrak{u}(c) - \mathfrak{u}(b))}{\theta(\mathfrak{u}(d) - \mathfrak{u}(b))\theta(\mathfrak{u}(c) - \mathfrak{u}(d))} \\
+&\frac{\theta(\vec{x} + \mathfrak{u}(c) + \mathfrak{u}(a))}{\theta(\vec{x} + \mathfrak{u}(c) + \mathfrak{u}(b))\theta(\mathfrak{u}(a) - \mathfrak{u}(b))}\frac{\theta(\mathfrak{u}(c) - \mathfrak{u}(a))}{\theta(\mathfrak{u}(a) - \mathfrak{u}(d))\theta(\mathfrak{u}(c) - \mathfrak{u}(d))} \\
=&\frac{\theta(\vec{x} + \mathfrak{u}(a) + \mathfrak{u}(b))}{\theta(\mathfrak{u}(a) - \mathfrak{u}(d))\theta(\vec{x} + \mathfrak{u}(d) + \mathfrak{u}(b))}\frac{\theta(\vec{x} + \mathfrak{u}(c) + \mathfrak{u}(d))}{\theta(\vec{x} + \mathfrak{u}(c) + \mathfrak{u}(b))\theta(\mathfrak{u}(d) - \mathfrak{u}(b))},
\end{aligned} \tag{B.4}$$

which can be rewritten as

$$\begin{aligned}&-K(\mathfrak{u}(b),\mathfrak{u}(a),-\vec{x}-\mathfrak{u}(d))K(\mathfrak{u}(d),\mathfrak{u}(b),\mathfrak{u}(c))\\&+K(\mathfrak{u}(b),\mathfrak{u}(a),-\vec{x}-\mathfrak{u}(c))K(\mathfrak{u}(d),\mathfrak{u}(a),\mathfrak{u}(c))\\&=K(\mathfrak{u}(b),\mathfrak{u}(d),-\vec{x}-\mathfrak{u}(c))K(\mathfrak{u}(d),\mathfrak{u}(a),-\vec{x}-\mathfrak{u}(b)),\end{aligned} \tag{B.5}$$

where

$$K(z,x,y)=\frac{\theta(y-x)}{\theta(z-x)\theta(y-z)}=-K(z,y,x). \tag{B.6}$$

Finally, we can rewrite this for the Kronecker function

$$F(z,x,\vec{\alpha})=\frac{\theta(\mathfrak{u}(z)-\mathfrak{u}(x)+\vec{\alpha})}{\theta(\mathfrak{u}(z)-\mathfrak{u}(x))\theta(\vec{\alpha})}\sum_{j=1}^{h}\partial_{v_j}\theta(v)\big|_{v=0}\omega_j(z) \tag{B.7}$$

as

$$F(b,d,\vec{\alpha})F(d,a,\vec{\beta})=F(b,a,\vec{\alpha})F(d,a,\vec{\beta}-\vec{\alpha})-F(b,a,\vec{\beta})F(d,b,\vec{\beta}-\vec{\alpha}), \tag{B.8}$$

provided that $(\vec{\alpha},\vec{\beta})=(-\vec{x}-\mathfrak{u}(c)-\mathfrak{u}(b),-\vec{x}-\mathfrak{u}(b)-\mathfrak{u}(d))$ for some vector $\vec{x}$ and point c.

We can also reach a more pleasant form that avoids negatives by relabeling $\vec{\beta}\mapsto\vec{\beta}+\vec{\alpha}$ so

$$F(b,a,\vec{\alpha})F(d,a,\vec{\beta})=F(b,d,\vec{\alpha})F(d,a,\vec{\alpha}+\vec{\beta})+F(b,a,\vec{\alpha}+\vec{\beta})F(d,b,\vec{\beta}) \tag{B.9}$$

provided that $(\vec{\alpha},\vec{\beta})=(\vec{x},\mathfrak{u}(c)-\mathfrak{u}(d))$ for some vector $\vec{x}$ and point c. One can notice that this identity is independent of the choice of basepoint for the Abel map, since this either gets cancelled by differences, or can be absorbed into $\vec{x}$.

C Pole-cancellation series

In order to make sure that the pole-cancellation function cancels the entire divergence from $1/\theta(\vec{\alpha})$ in the Kronecker function, we can relabel $P(\vec{\alpha})=\tilde{P}(\vec{\alpha})\theta(\vec{\alpha})$ so

$$\tilde{P}(\vec{\alpha})\theta(\vec{\alpha})F(z,x,\vec{\alpha})=\tilde{P}(\vec{\alpha})\frac{\theta(\mathfrak{u}(z)-\mathfrak{u}(x)+\vec{\alpha})}{\theta(\mathfrak{u}(z)-\mathfrak{u}(x))}\sum_{j=1}^{h}\partial_{v_j}\theta(\vec{v})\big|_{\vec{v}=0}\omega_j(z)=\sum_{\vec{n}\in\mathbb{Z}^h_{\geq 0}}g^{(\vec{n})}(z,x)\prod_{j=1}^{h}\alpha_j^{n_j}. \tag{C.1}$$

Writing $\sigma(z,x)=\sum_{j=1}^{h}\partial_{v_j}\theta(\vec{v})\big|_{\vec{v}=0}\omega_j(z)/\theta(\mathfrak{u}(z)-\mathfrak{u}(x))$ for the part containing the normalized pole, and expanding $\tilde{P}$ and θ as

$$\tilde{P}(\vec{\alpha})=\sum_{\vec{n}\in\mathbb{Z}^h_{\geq 0}}\tilde{P}^{(\vec{n})}\prod_{j=1}^{h}\alpha_j^{n_j}\quad;\quad\theta(\mathfrak{u}(z)-\mathfrak{u}(x)+\vec{\alpha})=\sum_{\vec{n}\in\mathbb{Z}^h_{\geq 0}}\theta^{(\vec{n})}(\mathfrak{u}(z)-\mathfrak{u}(x))\prod_{j=1}^{h}\alpha_j^{n_j}, \tag{C.2}$$

we can proceed to directly expanding the pole-cancelled Kronecker function

$$\tilde{P}(\vec{\alpha})\theta(\vec{\alpha})F(z,x,\vec{\alpha})=\sum_{\vec{m},\vec{n}\in\mathbb{Z}^h_{\geq 0}}\tilde{P}^{(\vec{m})}\theta^{(\vec{n})}(\mathfrak{u}(z)-\mathfrak{u}(x))\sigma(z,x)\prod_{j=1}^{h}\alpha_j^{m_j+n_j}, \tag{C.3}$$

so we recognize

$$g^{(\vec{n})}(z,x)=\sum_{\vec{k}\leq\vec{n}}\tilde{P}^{(\vec{k})}\theta^{(\vec{n}-\vec{k})}(\mathfrak{u}(z)-\mathfrak{u}(x))\sigma(z,x). \tag{C.4}$$

So, we can now choose to select the coefficients $\tilde{P}^{(\vec{k})}$ in the expansion of $\tilde{P}$ so as to manifest the properties we desire in our integration kernels g. With this choice, we may try to make is to select the residues of the kernels in a convenient way, such as by reducing how many kernels have residues to have fewer regularizations that need to be done. Since $\mathrm{res}_{z=x}\sigma(z,x) = 1$, we see that the residue of the kernels is

$$\mathrm{res}_{z=x} g^{(\vec{n})}(z,x) = \sum_{\vec{k}\leq\vec{n}} \tilde{P}^{(\vec{k})}\theta^{(\vec{n}-\vec{k})}(\vec{0}). \tag{C.5}$$

Since θ refers to the odd theta function, its even coefficients will vanish

$$\vec{n}-\vec{k}\text{ even} \implies \theta^{(\vec{n}-\vec{k})}(\vec{0}) = 0, \tag{C.6}$$

so by choosing $\tilde{P}$ to be an even function, we can immediately make the residues for all kernels with even indices vanish, since they contain terms $\tilde{P}^{(\vec{k})}\theta^{(\vec{n}-\vec{k})}$ where there is either odd $\vec{k}$ or even $\vec{n}-\vec{k}$.

$$\tilde{P}^{(\vec{k})} = 0 \;\forall\; \vec{k}\text{ odd} \implies \mathrm{res}_{z=x} g^{(\vec{n})}(z,x) = 0 \;\forall\; \vec{n}\text{ even} \tag{C.7}$$

We can then proceed to working with the remaining equations for even $\vec{n}$. We may fix the residues, and try to recursively find the necessary coefficients. For example, at genus one, we can fix $\mathrm{res}_{z=x} g^{(n)}(z,x) = \delta_{0n}$, and then we would find

$$\mathrm{res}_{z=x} g^{(1)}(z,x) = 1 = \tilde{P}^{(0)}\theta^{(1)}(0) \implies \tilde{P}^{(0)} = 1/\theta^{(1)}(0) \tag{C.8}$$

$$\mathrm{res}_{z=x} g^{(3)}(z,x) = 0 = \tilde{P}^{(2)}\theta^{(1)}(0) + \tilde{P}^{(0)}\theta^{(3)}(0) \implies \tilde{P}^{(2)} = -\frac{\theta^{(3)}(0)}{[\theta^{(1)}(0)]^2} \tag{C.9}$$

$$\vdots \tag{C.10}$$

which are precisely the coefficients of $\tilde{P}(\alpha) = \alpha/\theta(\alpha)$, which gives us $P(\alpha) = \tilde{P}(\alpha)\theta(\alpha) = \alpha$, as we would expect since that is exactly what is used (Equation 4.4).

However, at higher genus, the number of equations we fix grows faster than the number of coefficients we can use as parameters. For example, considering kernels up to weight $2k+1$, we will have k^2+k equations

$$\overbrace{\underbrace{g^{(1,0)}, g^{(0,1)}}_{2};\ \underbrace{g^{(3,0)},\ g^{(2,1)},\ g^{(1,2)},\ g^{(0,3)}}_{4};\cdots;\ \underbrace{g^{(2k+1,0)},\cdots,g^{(0,2k+1)}}_{2k}}^{k^2+k} \tag{C.11}$$

but these equations would only depend on coefficients $\tilde{P}^{(\vec{k})}$ up to weight $2k$, so we will have k^2 parameters

$$\overbrace{\underbrace{\tilde{P}^{(0,0)}}_{1};\ \underbrace{\tilde{P}^{(2,0)},\ \tilde{P}^{(1,1)},\ \tilde{P}^{(0,2)}}_{3};\cdots;\ \underbrace{\tilde{P}^{(2k,0)},\cdots,\tilde{P}^{(0,2k)}}_{2k-1}}^{k^2} \tag{C.12}$$

so there will not be a solution to the system unless k of the equations are not fixed[25]. As $k\to\infty$, we will have an infinite number of unfixed residues, and consequently infinitely many of our

[25]Technically, this depends on the algebraic independence of the derivatives of the theta function, which is challenging to show. Their independence for low weights can be seen numerically. Hopefully in further work, or upon finding a suitable reference, the proof given can be made more precise.

kernels will have non-vanishing residues. The same argument holds true for higher genera, as we will find that '# of equations' − '# of parameters' $\sim O(k^{h-1})$.

Seeing that there will be an infinte number of kernels with non-vanishing residues anyway, we will stick with the choice $\tilde{P}(\vec{\alpha}) = 1$ giving $P(\vec{\alpha}) = \theta(\vec{\alpha})$. In this case, we have a simple closed form for the kernels

$$g^{(\vec{n})}(z,x) = \theta^{(\vec{n})}(\mathfrak{u}(z) - \mathfrak{u}(x))\sigma(z,x). \tag{C.13}$$

D Combinatorics for the genus two Fay identity

Before analyzing the specific case involving different components of the Abel map, let us consider the simpler case where we are dealing only with a function $[f(z)]^n$ with $(\partial_z)^d$ acting on it. We end up having the d derivatives distributed between n terms, resulting in

$$(\partial_z)^d [f(z)]^n = \sum_{\substack{\vec{p} \in \mathbb{Z}^n_{\geq 0} \\ \sum_k p_k = n \\ \sum_k k p_k = d}} C(\vec{p}) \prod_{k=0}^{n} [f^{(k)}(z)]^{p_j} \tag{D.1}$$

where the vector $\vec{p}$ determines how the derivatives are distributed, with the kth component indicating how many powers of $f^{(k)}$ the term contains. The sum of all powers must be n, and the total number of derivatives must be d, giving the constraints on p in the sum. The coefficient $C(\vec{p})$ contains the combinatoric factors corresponding to the powers pulled down from power rules, as well as by combining like terms through different 'paths' the derivatives take.

Instead of trying to use power rules, all the combinatoric factors can be identified by labeling all of our derivatives and functions, and considering all combinations in which they may be assigned. For example, let us write $(\partial_z)^3[f(z)]^2$ as $\partial_1\partial_2\partial_3 f_1 f_2$. Then, the term corresponding to $\vec{p} = (0,1,1)$ can occur as

$$\begin{aligned} &(\partial_1\partial_2 f_1)(\partial_3 f_2), \quad (\partial_1\partial_3 f_1)(\partial_2 f_2), \quad (\partial_2\partial_3 f_1)(\partial_1 f_2), \\ &(\partial_3 f_1)(\partial_1\partial_2 f_2), \quad (\partial_2 f_1)(\partial_1\partial_3 f_2), \quad (\partial_1 f_1)(\partial_2\partial_3 f_2), \end{aligned} \tag{D.2}$$

giving us $\tilde{C}(0,1,1) = 6$. If we factor out $n!$ since the order of f's does not matter (corresponding to taking only the first row in Equation D.2), the combinatorics amount to dividing the derivatives into n groups, where p_j of the groups have size j. In the example, this corresponds to

$$\overbracket{\partial_1\partial_2}\partial_3, \quad \overbracket{\partial_1\partial_2\partial_3}, \quad \partial_1\overbracket{\partial_2\partial_3}. \tag{D.3}$$

The number of ways to make this selection is $\frac{d!}{(\prod_{k=1}^{d}(k!)^{p_k})(\prod_{k=1}^{d} p_k!)}$, where $d!$ is the number of ways to arrange the derivatives, the first term in the quotient ignores the permutations within the strings, and the second term in the quotient ignores the order of the strings. In total, this gives us $C(\vec{p}) = \frac{n!d!}{(\prod_{k=1}^{d}(k!)^{p_k})(\prod_{k=1}^{d} p_k!)}$. We correctly see that $C(0,1,1) = \frac{2!3!}{1!2!1!1!} = 6$, and can compare to the direct approach where one does

$$\partial^3 f^2 = 2\partial^2 f f' = 2\partial(f')^2 + 2\partial f f'' = 6f'f'' + \cdots. \tag{D.4}$$

Now, we apply this conclusion to terms of the form $\left(\frac{d}{dp}\right)^N \left(\prod_{j=1}^{h} f_j(p)^{n_j}\right)$. Before we may use $\vec{p}$ to decide the number of terms with each order of derivatives, we must choose how many derivatives apply for each j. This makes us write

$$\left(\frac{d}{dp}\right)^N \left(\prod_{j=1}^h f_j(p)^{n_j}\right) = \sum_{\substack{\vec{d}\,\in\,\mathbb{Z}^h_{\geq 0} \\ \sum_i d_i\,=\,N}} \prod_{j=1}^h \sum_{\substack{\vec{p}^j\,\in\,\mathbb{Z}^{d_j}_{\geq 0} \\ \sum_k p^j_k\,=\,n \\ \sum_k k p^j_k\,=\,d}} C(\vec{p}^j) \prod_{k=0}^n [f^{(k)}(z)]^{p_j}, \tag{D.5}$$

where $\vec{d}$ selects where the derivatives land, and we now have a vector $\vec{p}^j$ for each component. Now, we apply this conclusion to $\left(\frac{d}{dp}\right)^N \sum_{\vec{n}\in\mathbb{Z}^h_{\geq 0}} \left(\prod_{j=1}^h f_j(p)^{n_j}\right)$, so the derivative applies to each term with $\vec{n}$. Finally, we must identify what this means for our original expression

$$\left(\frac{d}{dp}\right)^N \sum_{\vec{n}\in\mathbb{Z}^h_{\geq 0}} \tilde{F}^{(\vec{M};\vec{n})}(a,b,c) \left(\prod_{j=1}^h \beta_j^{n_j}\right), \tag{D.6}$$

for $\beta_j(p) = \mathfrak{u}_j(p) - \mathfrak{u}_j(c)$, with us sending $p \mapsto c$ at the end. Since $\beta_j(c) = 0$, we find that the only terms that survive are those without terms that have no derivatives $p^j_0 = 0$. Since we only have N derivatives, terms with a larger number of powers would necessarily have a term without a derivative and vanish, so we have $\sum_{j=1}^h n_j \leq N$. The remaining terms will contain derivatives of β_j with respect to p, which by the definition of the Abel map must be the functional component of the corresponding holomorphic differential, so $\beta_j^{(n)} = \omega_j^{(n-1)}$ for $n \geq 1$.

Putting it all together, we find the expression

$$\sum_{\substack{\vec{n},\,\vec{d}\,\in\,\mathbb{Z}^h_{\geq 0} \\ \sum_i n_i\,\leq\,N \\ \sum_i d_i\,=\,N}} \left(\prod_{j=1}^h \sum_{\substack{\vec{p}^j\,\in\,\mathbb{Z}^{d_j}_{\geq 0} \\ \sum_k p^j_k\,=\,n_j \\ \sum_k k p^j_k\,=\,d_j \\ p^j_0\,=\,0}} \frac{(N!)(n_j!)}{\prod_{k=1}^{d_j} (k!)^{p^j_k} (p^j_k!)} [\omega_j(c)^{p^j_1} \omega_j'(c)^{p^j_2} \cdots]\right) \times \tilde{F}^{(\vec{m};\vec{n})}(a,b,c) = 0, \tag{D.7}$$

as in Equation 7.26.

E Quasiperiodicity in the auxiliary variable on the Schottky cover

We start by expressing the terms in F_j as differences of fractions,

$$F_j(z,x) = \sum_{\gamma\in\Gamma} dz \left(\frac{1}{z-\gamma x} - \frac{1}{z-\gamma P_j}\right) W(\gamma). \tag{E.1}$$

As we plug in $x \mapsto \gamma_k x$, one may naively assume that one may be able to relabel the sum on only the first term to absorb the new generator. However, each fraction independently is not Moebius invariant, and their partial sums are not absolutely convergent, which would make such a relabeling change the value of the form. Instead, we must insert additional terms to recover

Moebius invariant cross-ratios,

$$\begin{aligned} F_j(z,\gamma_k x) &= \sum_{\gamma\in\Gamma} dz \left(\frac{1}{z-\gamma\gamma_k x} - \frac{1}{z-\gamma P_j}\right) W(\gamma) \\ &= \sum_{\gamma\in\Gamma} dz \left(\frac{1}{z-\gamma\gamma_k x} - \frac{1}{z-\gamma x} + \frac{1}{z-\gamma x} - \frac{1}{z-\gamma P_j}\right) W(\gamma) \\ &= \sum_{\gamma\in\Gamma} dz \left(\frac{1}{z-\gamma\gamma_k x} - \frac{1}{z-\gamma x}\right) W(\gamma) + F_j(z,x). \end{aligned} \tag{E.2}$$

Now, we can use a similar strategy to identify the first term on the right hand side,

$$\begin{gathered} \sum_{\gamma\in\Gamma} dz \left(\frac{1}{z-\gamma\gamma_k x} - \frac{1}{z-\gamma x}\right) W(\gamma) = \\ \sum_{\gamma\in\Gamma} dz \left(\frac{1}{z-\gamma\gamma_k x} - \frac{1}{z-\gamma\gamma_k P_k} - \frac{1}{z-\gamma x} + \frac{1}{z-\gamma P_k}\right) W(\gamma) = \\ \sum_{\gamma\in\Gamma} dz \left(\frac{1}{z-\gamma\gamma_k x} + \frac{1}{z-\gamma\gamma_k P_k}\right) W(\gamma) - F_k(z,x) = F_k(z,x)(w_k^{-1}-1). \end{gathered} \tag{E.3}$$

Putting together the results, we find

$$F_j(z,\gamma k x) = F_k(z,x)(w_k^{-1}-1) + F_j(z,x), \tag{E.4}$$

as in Equation 8.4.

F Post-submission corrections

Here I go into a bit of detail on a few points where the original submission fell short.

I would like to thank Prof. Claude Duhr for bringing to my attention the question addressed in Section F.1, and Sven Stawinski for helping idenfity the key details to resolving it.

I would like to thank Ji Zhexian for bring to my attention the question addressed in Section F.2, and Egor Im for helping verify the solution.

F.1 Divisor of the theta Kronecker form

In this section, we consider more closely the divisor of the theta Kronecker form. In particular, one asks if it is indeed true that the theta Kronecker form (Equation 7.2) is defined appropriately for any choice of pole x. This starts with developing a better understanding of the differential

$$[\psi(z)]^2 = d_z(\theta_D(\mathfrak{u}(z)-\mathfrak{u}(p)))\Big|_{p=z} = \sum_{j=1}^{h} \partial_{v_j}\theta_D(\vec{v})\Big|_{v=\vec{0}} \omega_j(z), \tag{F.1}$$

where we use thet notation of $\psi(z)$ from Equation 2.16.

In footnote 14, we suggest that the differential $d_z(\theta_D(\mathfrak{u}(z)-\mathfrak{u}(p)))\Big|_{p=z}$ vanishes for $z\in D$ with $\mathfrak{u}(z)\neq 0$. This is true, and it indeed cancels the spurious poles produced by the theta function in the denominator of the theta Kronecker form. However, abelian differentials on a genus h surface must vanish for a divisor of degree $2h-2$ [Ber10, Lemma 4.1.2]. As a result, there are $h-1$ zeros not taken into account, which we will label with the divisor $\tilde{D}$. For an arbitrary divisor D, it is hard to predict where these remaining zeros end up. However, for an odd theta function, one finds that the divisor of $d_z(\theta_D(\mathfrak{u}(z)-\mathfrak{u}(p)))\Big|_{p=z}$ is precisely $2D$ [Ber10, Proposition 6.1.1], so $\tilde{D} = D$.

When x coincides with one of the zeros in $\tilde{D}$, the differential $[\psi(z)]^2$ cancels the pole that would have been produced. As a result, for any choice of characteristics there are $h-1$ points where the pole cannot be placed. Furthermore, if we are working with an odd theta function, we find for $x \in \tilde{D} = D$ that

$$\theta_D(\mathfrak{u}(z) - \mathfrak{u}(x)) = -\theta_D(\mathfrak{u}(x) - \mathfrak{u}(z)) = 0 \; \forall z, \tag{F.2}$$

so the theta function in the denominator identically vanishes and the Kronecker form is ill-defined. This behavior is related to the special divisors mentioned in Section 2.4.1, since one can show that theta functions corresponding to a special divisor identically vanish [Ber10, Corollary 5.2.3].

This degeneration of the theta Kronecker form may be related to the challenges relating the theta Kronecker forms to the Schottky Kronecker forms as in Section 9.1.3. Depending on the choice of the pole x, a basis of theta Kronecker forms that seemed valid may become degenerate. One must be careful in translating between the languages to circumvent this issue.

F.2 Formula for Schottky integration kernels

In Section 8.3.2, we gave a starting point for how the integration kernels generated by the Schottky Kronecker form can be calculated as an average over genus one integration kernels. The approach taken sought to isolate only the generators that could contribute to the words labeling the integration kernels, and then calculate the corresponding coefficients for each word. However, this approach leads to a cumbersome expression with a nested definition for the group (Equation 8.18), and misses the contributions that inverse generators may have (Equation 8.16). Here we present a more simple and efficient approach discovered after the fact, where one directly calculates coefficients for every element of the Schottky group, excluding non-contributing elements by finding vanishing coefficients rather than through an involved summation.

As in Section 8.3.2, we will amend our notations to focus on groups with consecutive indices. We will write our word and our Schottky element in a way that groups repeated entries,

$$\text{word} = b_{i_1}^{n_1} \cdots b_{i_s}^{n_s} \tag{F.3}$$

$$\gamma = \gamma_{j_1}^{m_1} \cdots \gamma_{j_l}^{m_l} \tag{F.4}$$

where s and l give us the number of groups, $n_k > 0$ give us the lengths of the groups for the words, and $m_k \neq 0$ give us the lengths of the groups for the elements. In particular, one is allowed to have $m_k < 0$, to account for inverses that appear in the element of the Schottky group. This notation is made unique by forcing $i_k \neq i_{k+1}$ and $j_k \neq j_{k+1}$, so that the groups are always of the largest size possible, and so that $\gamma_j \gamma_j^{-1}$ never appears in our element.

We can expand the genus one Kronecker form

$$F(\gamma^{-1}z, x, w_j|\Gamma_j)b_j = \sum_{k=0}^{\infty} b_j^k g^{(k)}(\gamma^{-1}z, x|\Gamma_j). \tag{F.5}$$

Thus, when finding the coefficient of $b_{i_1}^{n_1} \cdots b_{i_s}^{n_s}$ in $W(\gamma)F(\gamma^{-1}z, x, w_j|\Gamma_j)b_j$, if $j = i_s$, some contributions may come from using higher weight genus one kernels,

$$\omega_{i_1 \cdots i_s j}(z, x|\Gamma) b_{i_1}^{n_1} \cdots b_{i_s}^{n_s} = \sum_{\gamma \in \Gamma/\Gamma_j} \sum_{k=0}^{\delta_{j i_s} n_s} W(\gamma) b_{i_s}^{k} g^{(k)}(\gamma^{-1}z, x). \tag{F.6}$$

What remains is for us to find the coefficient of $b_{i_1}^{n_1} \cdots b_{i_s}^{n_s}$ in $W(\gamma)$. We can do this recursively, writing

$$C(b_{i_1}^{n_1} \cdots b_{i_s}^{n_s}, \gamma_{j_1}^{m_1} \cdots \gamma_{j_l}^{m_l}) = \begin{cases} s = 0: & 1 \quad \text{(empty word)}, \\ s \neq 0 = l: & 0 \quad \text{(non-empty word, identity element)}, \\ i_1 \neq j_1: & C(b_{i_1}^{n_1} \cdots b_{i_s}^{n_s}, \gamma_{j_2}^{m_2} \cdots \gamma_{j_l}^{m_l}) \quad \text{(first letters don't match)}, \\ i_1 = j_1: & \sum_{k=0}^{n_1} \frac{(m_1)^k}{k!} C(b_{i_1}^{n_1 - k} \cdots b_{i_s}^{n_s}, \gamma_{j_2}^{m_2} \cdots \gamma_{j_l}^{m_l}) \quad \text{(first letters match)}, \end{cases} \tag{F.7}$$

where the recursive definition parses one group of generators at a time, using the base cases for empty words and elements.

Then, we finally find

$$\omega_{i_1 \cdots i_s j}(z, x|\Gamma) = \sum_{\gamma \in \Gamma/\Gamma_j} \sum_{k=0}^{\delta_{j i_s} n_s} C(b_{i_1}^{n_1} \cdots b_{i_s}^{n_s - k}, \gamma) g^{(k)}(\gamma^{-1}z, x). \tag{F.8}$$

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