UC DAVIS
DEPARTMENT OF PHYSICS AND ASTRONOMY

Geometry of non-planar on-shell diagrams

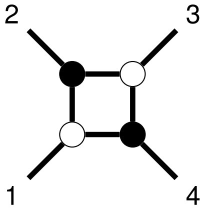
Artyom Lisitsyn
Based on ongoing work with U. Oktem, M. Sherman-Bennett, J. Trnka

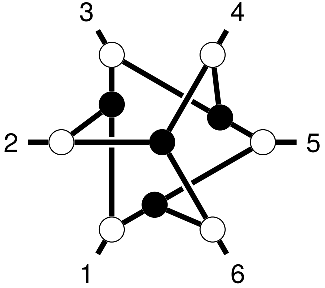
UC DAVIS

On-shell diagrams

- Diagrammatic representation of scattering amplitudes in $\mathcal{N} = 4$ SYM
- Gluing together of 3-point amplitudes
  - White vertex : $\lambda_1 \propto \lambda_2 \propto \lambda_3$
  - Black vertex : $\tilde{\lambda}_1 \propto \tilde{\lambda}_2 \propto \tilde{\lambda}_3$
- Amplitude is related to an on-shell form computed by boundary measurements
- The constraints can be encoded in an integral over $G(k, n)$

$$\Omega = \frac{\mathrm{d}^{k\times n}C}{\mathrm{vol}(GL(k))}\tilde{f}(C)\delta^{k\times 4}(C\cdot\tilde{\eta})\delta^{k\times 2}(C\cdot\tilde{\lambda}).$$



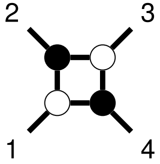


Planar diagrams and corresponding form

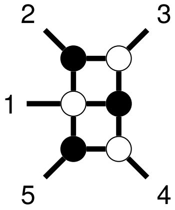
pole structure of on-shell form $\longleftrightarrow$ boundaries of positive Grassmannian

$$\tilde{f}(C) = \frac{1}{(1\cdots k)(2\cdots k+1)\cdots(n\cdots k-1)}$$

$$G_+(k,n) = \{C \in G_{\mathbb{R}}(k,n)\,|\,(i_1\cdots i_k) > 0\,\forall\, i_1 < \cdots < i_k\}$$



$$PT(1,2,3,4) = \frac{1}{(12)(23)(34)(41)}$$



$$PT(1,2,3,4,5) = \frac{1}{(12)(23)(34)(45)(51)}$$

Combinatorics for MHV diagrams

A counting argument shows that (*Arkani-Hamed et al. 2015*)

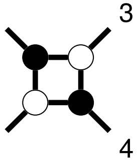
- There are $n-2$ trivalent black vertices
- Each black vertex connects to three external edges (through white vertices)

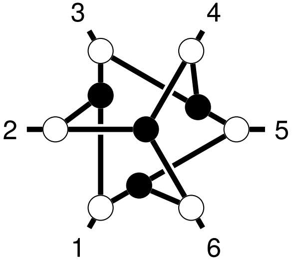
$$\Downarrow$$

Every MHV diagram is described by $n-2$ triplets

For example,

$$T_4 = \{(1,2,3),(1,3,4)\}$$
$$T_6 = \{(1,2,3),(3,4,5),(5,6,1),(2,4,6)\}$$





MHV diagrams and corresponding form

The form can be computed as

$$f_T = \prod_{(i,j,k)\in T} \left(\frac{1}{(ij)(jk)(ki)} \right) \delta^{2\times 4}(C \cdot \tilde{\eta}) \delta^{2\times 2}(C \cdot \tilde{\lambda})$$

We would like to write this in standard form as $f_T = \tilde{f}_T \times \delta^{2\times 4}(\lambda \cdot \tilde{\eta})\delta^{2\times 2}(\lambda \cdot \tilde{\lambda})$,

$\swarrow$ $\searrow$

Gauge-fixing $C^{\perp}$ to match $\lambda^{\perp}$

$$\tilde{f}_T = \frac{(\det M_{ab}/(ab))^2}{\prod_{(i,j,k)\in T}(ij)(jk)(ki)}$$

Keeping track of constraints throughout the amalgamation

$$\tilde{f}_T = \sum_{\sigma \in \hat{S}_n} PT(\sigma)$$

(I'll review these formulas in later slides)

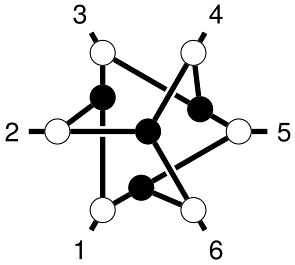
Outline

- Determinantal formula
  - Theorem on conditions for equivalent forms
  - Square moves and sphere moves
  - Factorization of the form
  - Doublets
- Decomposition formula
  - Positive regions
  - Triangulation of pseudo-positive geometry
  - Theorem about connectedness of result

Determinantal formula

The matrix $C^{\perp}(\vec{\alpha}^*)$ has one row per triplet, with entries (jk) in column i.

$$C^{\perp}(\vec{\alpha}^*) = \begin{pmatrix} (23) & (31) & (12) & 0 & 0 & 0 \\ 0 & 0 & (45) & (53) & (34) & 0 \\ (56) & 0 & 0 & 0 & (61) & (15) \\ 0 & (46) & 0 & (62) & 0 & (24) \end{pmatrix}$$



$$T = \{(1,2,3),(3,4,5),(5,6,1),(2,4,6)\}$$

For any choice of columns $\{a, b\}$, the matrix M_{ab} is $C^{\perp}(\vec{\alpha}^*)$ with those columns removed.

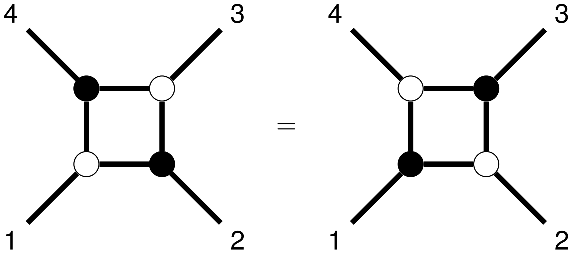
$$\tilde{f}_T = \frac{(\det M_{ab}/(ab))^2}{\prod_{(i,j,k)\in T}(ij)(jk)(ki)} = \frac{((53)(61)(24)+(34)(15)(46))^2}{(12)(23)(31)(34)(45)(53)(56)(61)(15)(24)(46)(62)}$$

Theorem on conditions for equal forms

Let T and T' be two sets of triplets. The following statements are equivalent:

1. The sets of triplets are related by a sequence of sphere moves.
2. The corresponding forms are equal: $f_T = f_{T'}$.
3. The corresponding doublets are the same: $D(T) = D(T')$.

Square move

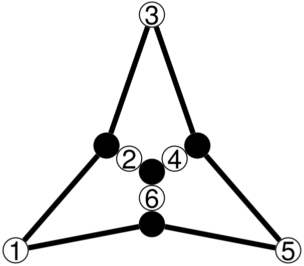


$$f_{\{(1,2,3),(1,3,4)\}} = f_{\{(2,1,4),(2,3,4)\}} = \frac{1}{(12)(23)(34)(41)}$$

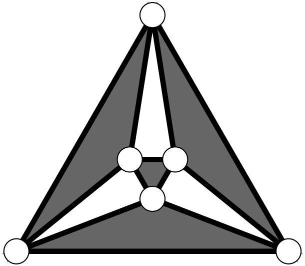
Sphere move

Two sets of triplets are related by a sphere move if the union of the two sets gives a triangulation of a sphere.

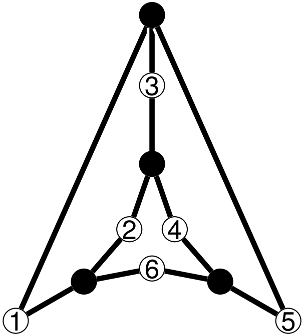
e.g. 6-point triangulation of octahedron



$\leftrightarrow$



$\leftrightarrow$

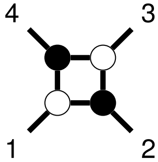


$$f_{\{(1,2,3),(3,4,5),(5,6,1),(2,4,6)\}} = f_{\{(2,3,4),(4,5,6),(6,1,2),(1,3,5)\}}$$

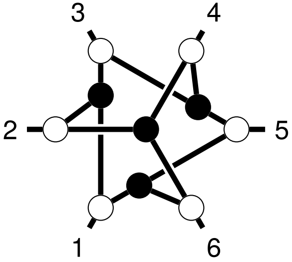
Doublets

The set of doublets corresponding to a set of triplets is

$$D(T) = \{(i,j) \,|\, i \text{ and } j \text{ appear together in an odd number of triplets in } T\}.$$



$$, \quad \begin{aligned} T &= \{(1,2,3),(1,3,4)\}, \\ D(T) &= \{(1,2),(2,3),(3,4),(4,1)\} \end{aligned}$$



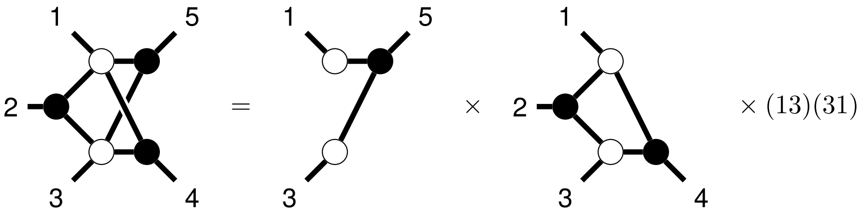
$$, \quad \begin{aligned} T &= \{(1,2,3),(3,4,5),(5,6,1),(2,4,6)\}, \\ D(T) &= \left\{ \begin{array}{c} (1,2),(2,3),(3,1),(3,4),(4,5),(5,4), \\ (5,6),(6,1),(1,5),(2,4),(4,6),(6,2) \end{array} \right\} \end{aligned}$$

Factorization

Suppose $R \subset T$ is a subset of triplets that itself corresponds to some diagram, and has at least one index not present elsewhere in T.
With $\{a_1, \cdots, a_r\}$ as the indices R shares with $T \setminus R$,

$$\tilde{f}_T = \tilde{f}_R \times \tilde{f}_{(T\setminus R)\cup P} \times \prod_{i=1}^{r}(a_i a_{i+1}),$$

where $P = \{(a_1, a_2, a_3), (a_1, a_3, a_4), \cdots, (a_1, a_{r-1}, a_r)\}$.
A diagram without such a subset R is called irreducible.



Factorization

Suppose $R \subset T$ is a subset of triplets that itself corresponds to some diagram, and has at least one index not present elsewhere in T.
With $\{a_1, \cdots, a_r\}$ as the indices R shares with $T \setminus R$,

$$\tilde{f}_T = \tilde{f}_R \times \tilde{f}_{(T \setminus R) \cup P} \times \prod_{i=1}^{r} (a_i a_{i+1}),$$

where $P = \{(a_1, a_2, a_3), (a_1, a_3, a_4), \cdots, (a_1, a_{r-1}, a_r)\}$.
A diagram without such a subset R is called irreducible.

Sketch of proof: Choosing the columns $\{a, b\}$ in the determinantal formula to be in R, the matrix is block triangular

$$\det(M_{T,ab}) = \det \begin{pmatrix} M_{R,ab} & 0 \\ M_{(T \setminus R)|_{\in R},ab} & M_{(T \setminus R)|_{\notin R}} \end{pmatrix} = \det(M_{R,ab}) \times \det\big(M_{(T \setminus R) \cup P, ij}\big) \times (\cdots)$$

Theorem on conditions for equal forms

Let T and T' be two sets of triplets. The following statements are equivalent:

1. The sets of triplets are related by a sequence of sphere moves.
2. The corresponding forms are equal: $f_T = f_{T'}$.
3. The corresponding doublets are the same: $D(T) = D(T')$.

Sketch of proof:
$(1) \implies (2)$ known from determinantal formula
(Cachazo et al. 2019; Castravet and Tevelev 2013)
$(2) \implies (3)$ is true for irreducible diagrams, and thus true for all through factorization
$(3) \implies (1)$ can be checked by studying the Euler characteristic of the manifold with triplets as faces and doublets as edges

Decomposition formula

$$\tilde{f}_T = \sum_{\sigma \in \hat{S}_n} PT(\sigma) = \sum_{\sigma \in \hat{S}_n} \frac{1}{(\sigma_1 \sigma_2) \cdots (\sigma_n \sigma_1)},$$

where $\hat{S}_n$ is the set of permutations up to cyclic shifts such that each triplet is ordered.

e.g. $T = \{(1,2,3),(1,3,4),(1,3,5)\} \longrightarrow \hat{S}_n = \{12345, 12354\}$

$$\tilde{f}_T = \frac{1}{(12)(23)(34)(45)(51)} + \frac{1}{(12)(23)(35)(54)(41)} = \frac{(31)}{(35)(51)(12)(23)(34)(41)}.$$

Positive regions

Let us define a *positive region* with $\varepsilon_i = \pm 1$ and $\{a_1, \cdots, a_n\} = \{1, \cdots, n\}$

$$PR(\varepsilon_1 a_1, \cdots, \varepsilon_n a_n) = \{C \in G_{\mathbb{R}}(2,n) \,|\, \varepsilon_i \varepsilon_j (a_i a_j) > 0\}.$$

e.g. $G_+(2,n) = PR(+1, \cdots, +n)$.

Twisted cyclicity:

$$PR(a_1, \cdots, a_n) = PR(-a_n, a_1, \cdots, a_{n-1})$$

Each equivalence class has a unique representative written as $PR(1, \cdots)$.

Codim-1 connectedness:

$$PR(\cdots, i, j, \cdots)\big|_{(ij)=0} = PR(\cdots, j, i, \cdots)\big|_{(ij)=0}$$

Canonical form: The canonical form of $PR(\pm a_1, \cdots, \pm a_n)$ is $PT(a_1, \cdots, a_n)$.

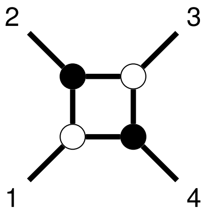
Pseudo-positive geometry for non-planar MHV diagrams

The on-shell form for an MHV diagram with triplets T is a canonical form of the pseudo-positive geometries defined by

$$G = \bigcup_{\sigma \in \hat{S}_n} PR(\varepsilon_{\sigma,1}\sigma_1, \cdots, \varepsilon_{\sigma,n}\sigma_n),$$

for arbitrary choices of $\varepsilon_{\sigma,i}$.

Choices for orientations : 4-point example

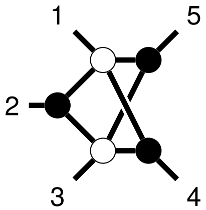


$$T = \{(1,2,3),(1,3,4)\} \longrightarrow G = PR(1,2,3,4) = G_+(2,4)$$

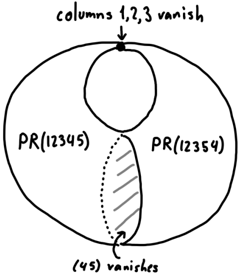
$$\tilde{T} = \{(1,2,3),(1,4,3)\} \longrightarrow \tilde{G} = PR(1,2,4,3) \cup PR(1,4,2,3)$$

$$f_T = PT(1,2,3,4) = -[PT(1,2,4,3) + PT(1,4,2,3)] = -f_{\tilde{T}}$$

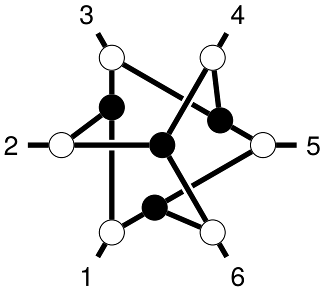
5-point example



$$T = \{(1,2,3),(1,3,4),(1,3,5)\} \longrightarrow G = PR(1,2,3,4,5) \cup PR(1,2,3,5,4)$$

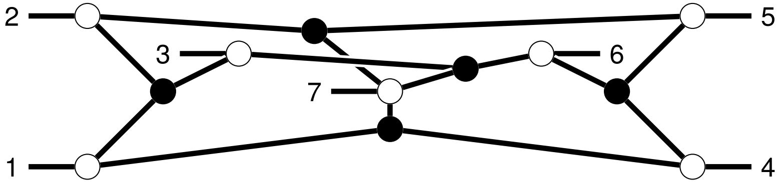


6-point example



$$T = \{(1,2,3),(3,4,5),(5,6,1),(2,6,4)\} \longrightarrow G = PR(1,4,2,5,3,6) \cup PR(1,4,2,5,6,3)$$
$$\cup\, PR(1,4,5,2,3,6) \cup PR(1,4,5,2,6,3)$$
$$\cup\, PR(1,2,5,3,6,-4) \cup PR(1,2,5,6,3,-4)$$
$$\cup\, PR(1,5,2,3,6,-4) \cup PR(1,5,2,6,3,-4).$$

7-point example



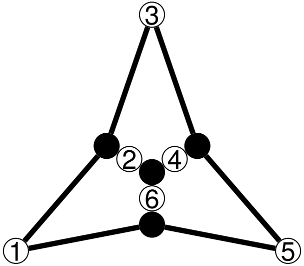
$$T = \{(1,2,3),(4,5,6),(1,7,4),(2,7,5),(3,7,6)\}$$

$$\begin{aligned} G = &PR(1,2,3,7,4,5,-6) \cup PR(1,6,2,3,7,4,-5) \cup PR(1,2,6,3,7,4,-5) \cup PR(1,5,6,2,3,7,4) \cup PR(1,5,2,6,3,7,4) \\ &\cup PR(1,2,3,7,5,6,4) \\ &\cup PR(1,2,3,7,6,4,-5) \cup PR(1,5,2,3,7,6,4) \\ &\cup PR(1,2,7,4,5,6,3) \\ &\cup PR(1,2,7,5,6,3,4) \cup PR(1,2,7,5,6,4,3) \\ &\cup PR(1,2,7,6,3,4,-5) \cup PR(1,2,7,6,4,3,-5) \cup PR(1,5,2,7,6,3,4) \cup PR(1,2,7,6,4,5,3) \cup PR(1,5,2,7,6,4,3) \\ &\cup PR(1,7,4,5,2,6,3) \cup PR(1,7,4,5,6,2,3) \\ &\cup PR(1,7,5,2,6,3,4) \cup PR(1,7,5,2,6,4,3) \cup PR(1,7,5,6,2,3,4) \cup PR(1,7,5,6,2,4,3) \cup PR(1,7,5,6,4,2,3) \\ &\cup PR(1,7,6,4,5,2,3) \end{aligned}$$

Theorem on connectedness for internally planar diagrams

For any internally planar diagram, there exists an associated codim-1 connected geometry. This geometry is identified with the orientation of triplets induced by a planar embedding of the graph's internal edges.

e.g. for 6-point graph the connected decomposition can be found by reading the black vertices counterclockwise



$$T = \{(1,2,3),(3,4,5),(5,6,1),(2,6,4)\}$$

$$\Longrightarrow \hat{S}_n = \{142536, 142563, 145236, 145263, 125364, 125634, 152364, 152634\}$$

Sketch of proof

$$T = \{(1,2,3),(3,4,5),(5,6,1),(2,6,4)\}$$
$$\hat{S}_n = \{142536, 142563, 145236, 145263,$$
$$125364, 125634, 152364, 152634\}$$

Goal: Show that the regions are connected by swapping adjacent indices.
Step 1: Divide $\hat{S}_n$ into subsets in which triplets have defined orderings:

$$\{142536, 142563, 145236, 145263\} \leftrightarrow \{1 \prec 2 \prec 3, 4 \prec 5 \prec 3, 1 \prec 5 \prec 6, 4 \prec 2 \prec 6\},$$
$$\{125364, 125634, 152364, 152634\} \leftrightarrow \{1 \prec 2 \prec 3, 5 \prec 3 \prec 4, 1 \prec 5 \prec 6, 2 \prec 6 \prec 4\}.$$

Step 2: Connectedness within each poset by swapping adjacent indices.

$\sigma\tilde{\sigma}^{-1} \neq 1 \implies \exists(\sigma\tilde{\sigma}^{-1})_i > (\sigma\tilde{\sigma}^{-1})_{i+1} \implies \sigma_i \not\prec\not\succ \sigma_{i+1} \implies$ valid swap to bring the two closer

Step 3: Connectedness between posets has a bijection to a similar problem for perfect matchings of planar graphs. That problem is solved by *Propp 2002*.

Review & Outlook

- Equivalent on-shell forms are related by sphere moves,
  and are characterized by having the same doublets.
- On-shell forms are canonical forms of a large family of pseudo-positive geometries.
- For any internally planar diagram, there exist special geometries that are strongly connected.

- Are there any more properties that may help single out geometries,
  especially for the fully non-planar diagrams?
- What lessons can one learn to apply to beyond MHV?

References

- Arkani-Hamed, Nima et al. (2015). "On-Shell Structures of MHV Amplitudes Beyond the Planar Limit". In: *JHEP* 06, p. 179. DOI: 10.1007/JHEP06(2015)179. arXiv: 1412.8475 [hep-th].
- Cachazo, Freddy et al. (Feb. 2019). "Δ-algebra and scattering amplitudes". In: *Journal of High Energy Physics* 2019.2. ISSN: 1029-8479. DOI: 10.1007/jhep02(2019)005. URL: http://dx.doi.org/10.1007/JHEP02(2019)005.
- Castravet, Ana-Maria and Jenia Tevelev (2013). "Hypertrees, projections, and moduli of stable rational curves". In: *Journal für die reine und angewandte Mathematik (Crelles Journal)* 2013.675, pp. 121–180. DOI: doi:10.1515/CRELLE.2011.189. URL: https://doi.org/10.1515/CRELLE.2011.189.
- Propp, James (2002). "Lattice structure for orientations of graphs". In: arXiv: math/0209005 [math.CO]. URL: https://arxiv.org/abs/math/0209005.